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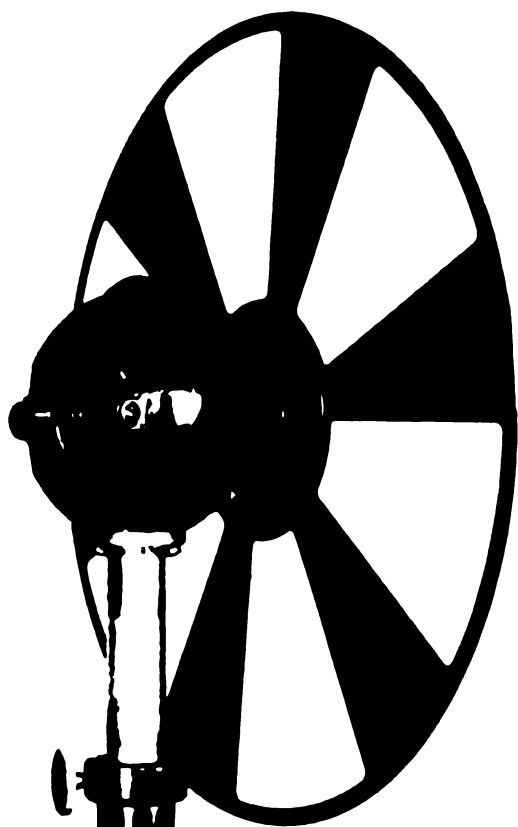
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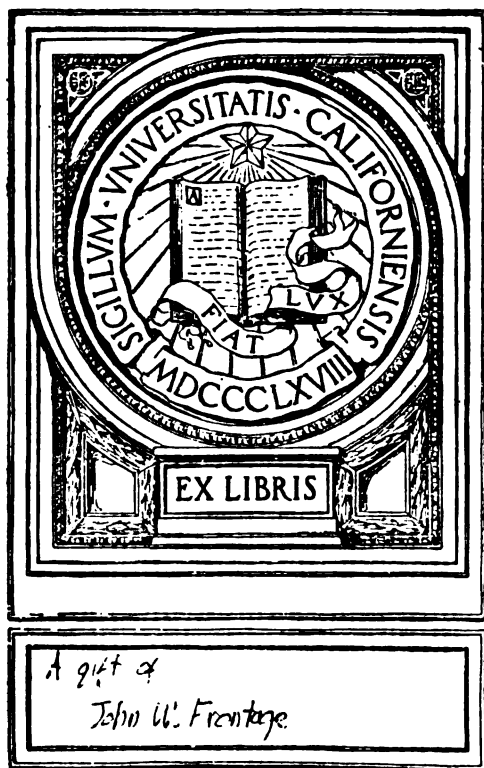
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DEPARTMENT OF COMMERCE AND LABOR

Vol. 2

BULLETIN

OF THE

BUREAU OF STANDARDS

S. W. STRATTON, DIRECTOR

JUNE, 1906

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TALBOT'S LAW AS APPLIED TO THE ROTATING SECTORED DISK.

By Edward P. Hyde.

1. Introduction.
2. Theoretical Discussion.
 - a. Radiation from a cylindrical source.
 - b. Method of mean distances.
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 - a. General method.
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 - d. Details of methods.
 - White light.
 - Colored light.
4. Experimental Results.
 - a. Effect of speed.
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1. INTRODUCTION.

The rotating sector disk is one of the most valuable adjuncts in photometric investigation, and yet from the first announcement by Talbot¹ of the law governing its operation until the present time the application of the sector disk has been limited by a widespread doubt of the general truth of the law announced by Talbot. This law is stated by Helmholtz as follows:² "If any part of the retina is excited with intermittent light, recurring periodically and

¹ Phil. Mag., Ser. 3, Vol. 5, p. 321; 1834.

² Physiolog. Optik, II Auflage, p. 483.

regularly in the same way, and if the period is sufficiently short, a continuous impression will result, which is the same as that which would result if the total light received during each period were uniformly distributed throughout the whole period."

Talbot's law is thus a statement of physiological rather than of physical phenomena, and depends for its explanation on the action of the eye. : Talbot recognized this and was consequently led to state, "I need hardly observe that it would be illogical to assert *a priori* the existence of this law of optics, however simple and natural it may appear, unless we were perfectly well acquainted with the circumstances which accompany the action of light upon the retina, which is very far from being the case. Its proof can rest on experiment alone, and by that it seems to be most satisfactorily established."

Many experimenters since Talbot have investigated the law, notably Plateau,¹ Helmholtz,² Fick,³ Kleiner,⁴ Wiedemann and Messerschmidt,⁵ and more recently Ferry,⁶ and Lummer and Brodhun.⁷ Of the earlier investigators Plateau, Helmholtz, Kleiner, and Wiedemann and Messerschmidt verified the law within their range of experimental error, which was always several per cent. Fick, from theoretical considerations of the complex action of the eye as shown in the phenomena known as "Anklingen" and "Abklingen," concluded that a law as simple as that of Talbot is impossible. He, moreover, repeated Plateau's experiments and was led to the conclusion that Talbot's law is not true in general, but that with more intense illuminations the action of the intermittent light is stronger than it should be according to the law, and that perhaps with very weak illumination the effect is reversed. Aubert⁸ criticised Fick's conclusions on the ground that the deviations which he found were of the same order of magnitude as his experimental error, and that therefore his results verified the law to within the limit of accuracy of his experiments.

¹ Pogg. *Annalen der Physik*, **85**, p. 457; 1835.

² *Physiolog. Optik*, II Auflage, p. 483.

³ Reichert's u. du Bois-Reymond's *Archiv.*, p. 739; 1863.

⁴ *Pflüger's Archiv*, **18**, p. 542; 1878.

⁵ *Wied. Ann.* **84**, p. 465; 1888.

⁶ *Phys. Rev.* **1**, p. 338; 1893.

⁷ *Zs. für Instrumentenkunde*, **16**, p. 299; 1896.

⁸ *Physiologie der Netzhaut*, p. 351.

Of the two more recent investigations Ferry verified the law for white light, but found quite large errors when the light transmitted through the rotating sector-disk was of a bluer quality than that incident on the other side of the photometer screen. His method consisted in mounting two sources of light on the photometer bench, one at each end, and interposing in the path of the rays from one source a rotating sector-disk with variable openings. The disk was mounted on a movable support so that it could readily be introduced in the path of the rays and quickly removed again after a reading had been made.

In this way, using as sources two 16 cp incandescent electric lamps of approximately the same color, he found the law to be verified for all openings of the sector down to a total angle of 24° . Moreover, as the voltage on the lamps was changed from 70 to 120 volts, provided the two lamps were changed together, the result was the same as before. He then mounted a 50 cp incandescent lamp at the disk end of the bar and a 16 cp lamp at the other end, and found as before that Talbot's law is verified if the two lamps are of approximately the same color. If, however, the 50 cp lamp was burned at a voltage much higher than the normal, or if a piece of slightly tinted blue glass was interposed in the path of the rays intercepted by the sector-disk, a very appreciable deviation from Talbot's law was introduced when the opening of the disk was less than 180° , the deviation increasing as the opening was made smaller. Using successively an enclosed arc lamp and a lime light on the sector-disk side of the photometer and an incandescent lamp on the other side, he obtained deviations amounting to 15 per cent and 10 per cent, respectively, at 24° , the deviation in each case being approximately zero when the opening of the disk was greater than 180° . In all cases in which deviations from the law were found the disk let through less light than that demanded by Talbot's law.

As a result of his observations Ferry concluded that although Talbot's law is true for white light, "with mixed light containing elements of different luminosity shining upon the retina, a rotating sector-disk will appear to not cut off all the elements in equal proportion, but will intercept most strongly the elements of low luminosity." Further, "with any given light the error introduced

by the use of the rotating sector disk increases as the aperture of the disk diminishes," although "with ordinary illuminants, the error is negligible when the total aperture of the disk is more than one-half the entire disk, but rapidly increases as this aperture is diminished."

Inasmuch as experiments made by the writer yielded results quite different from those of Ferry, it seems desirable to examine carefully his experiments and results. Unfortunately, owing to lack of detail in Ferry's paper, it is difficult to subject these to a searching criticism, and yet there are several points to which it may be well to call attention.

First, it is to be noticed that when the two incandescent lamps were varied simultaneously from 70 to 120 volts—i. e., over quite a large range of color—no error was observed, but that when one lamp was burned at a voltage somewhat higher than normal while the other was kept at the normal voltage a relatively large error was found. The question immediately arises: What would have been the result if the second lamp also had been burned at a voltage somewhat higher than normal? Since there is nothing in the paper to indicate the normal voltages of the two lamps, which were varied from 70 to 120 volts, we are at a loss to answer the question. It is probable, however, that the upper limit of such a large range would represent a voltage at least as high, if not higher, than the normal. On this assumption we would conclude that although the change in color produced by a decrease in voltage from 120 to 70 volts is accompanied by no error in Talbot's law, the color change due to a rise in voltage somewhat above the normal causes a relatively large error in the law for openings of small angles. In other words, the color corresponding to the normal voltage is a critical color.

Since this is quite improbable, we are led to conclude that the cause of the error when the lamp at the disk end of the bar was raised to a voltage somewhat higher than the normal, while the other lamp was kept at the normal voltage, is to be found in the color difference which existed on the two sides of the screen. While it is difficult to understand why the color of one side should influence the effect of the rotating sector on light from the other lamp, it is to be noted that in all probability a different form of photometer was used when the color difference existed, and this

might perhaps account for the error. Ferry states in his introduction that for lights of the same color a Lummer-Brodhun photometer was used, but that when a considerable color difference existed a Bunsen photometer was employed. When the difference in color on the two sides was small, he found the Nichols-Ritchie photometer to give the best results. In describing his experiments he does not state explicitly, however, which form of photometer he used in each case.

Because of this lack of definiteness in Ferry's paper it seemed very desirable to make further experiments on the influence of color on the action of the rotating sector'd disk. There are also two very definite reasons why Ferry's results are open to criticism. First, since the absolute illumination of the photometer screen was always much greater without the disk than with it, the Purkinje effect would produce errors in the results when the two sources differed to any extent in color. Secondly, the inverse square law applies rigorously only to point sources and can not be assumed *a priori* for such an extended source as the filament of an incandescent lamp, particularly



Fig. 1.—*Lummer and Brodhun's Arrangement.*

when it is surrounded by the reflecting and refracting glass bulb, which alters the curvature of the emitted light to such an extent that in some directions sharp images of the filament are formed several meters away. The lime light and the enclosed arc lamp, apart from the possible errors due to the inapplicability of the inverse square law, would seem to be unsuitable for accurate work because of their great unsteadiness.

The most satisfactory work that has been done on Talbot's law is that of Lummer and Brodhun at the Physikalisch-Technische Reichsanstalt. These investigators first verified the law for white light down to 50° total opening, by using two incandescent electric lamps, thus far being merely a repetition of Ferry's work. Recognizing, however, the possible error due to the inapplicability of the inverse square law, they modified their experiments in the following manner. Three incandescent lamps were mounted on the photometer bench, as shown in Fig. 1. Lamp *a* burned continuously but lamp

b and lamp *c* were burned successively, and the current through each was regulated until each produced approximately the same illumination of the screen. They were then both burned at the same time, the rotating disk with a total opening of 180° was placed between them and the photometer, and a balance was obtained by moving lamp *a*. In this way, since the distance from lamp *a* to the screen remained approximately constant, errors in the application of the inverse square law entered only as second order effects and were hence negligible. In a similar way each lamp in turn was burned with the disk set at 180° total opening, and then both were burned simultaneously with the disk set at 90° . In every case the deviations from the law were found to be of the same magnitude as the experimental error, which was less than one-half per cent. It is not stated, however, how far the process was carried, except that it was not extended to very small angular openings.

Although the results of Lummer and Brodhun are conclusive as far as they go, attention should be called to the fact that because the observational error was less than one-half per cent, it does not necessarily follow that Talbot's law was verified to such a high accuracy for all angular openings that they used. Since their method of observation consisted in verifying the law step by step, between 360° and 180° , 180° and 90° , etc., the error between 360° and, say, $22\frac{1}{2}^\circ$ would be distributed over four intervals, so that though the error within each interval may be less than one-half per cent it is possible that an error of 1 or 2 per cent might exist between 360° and $22\frac{1}{2}^\circ$ without being detected. This fact, together with a desire to check Ferry's results for colored light, led the writer to make the experiments described in the following pages.

In the investigation of Talbot's law by the use of the rotating sector disk some other law of the variation of the intensity of illumination must be assumed. Of the several possible ways of varying the intensity of illumination of the screen the method of varying the distance between the source and the screen is the simplest and most satisfactory, but presupposes the knowledge of the law of variation of the intensity of illumination with the distance for the source to be used. If the source is a point source the inverse square law holds; if the source is not a point source—and no actual source is strictly a point source—it must either be shown that within the limits of accuracy of the experiments the deviations from the inverse square

and uniform specific light intensity i . The problem then consists in determining the law of variation of the intensity of illumination with the distance for a uniform radiating cylinder.

Let us further assume Lambert's cosine law for the radiating cylinder. Although this law holds rigorously for black bodies only, the resulting error in the comparison of illuminations at different distances would be quite small. If, then, we let ϕ (Fig. 2) be the angle of emission for any element of surface dS of the radiating cylinder; θ the angle of incidence of any ray on a screen at P placed at right angles to $\overline{OP}$, where $\overline{OP}$ lies in the plane perpendicular to the axis of the cylinder at its middle point, O ; r the distance from the element dS to the screen at P ; then the intensity of illumination of the screen at P is

$$J = \int \int \frac{i \cos \phi \cos \theta dS}{r^2} \quad (1)$$

taken over that part of the curved surface of the cylinder convex toward P .

Expressing all the quantities involved in the above equation in terms of the two cylindrical coordinates a and y , we get for the intensity of illumination at P , at a distance l , from the axis of the cylinder—

$$J = i a \int \int \frac{(l \cos a - a)(l - a \cos a)}{(a^2 + l^2 - 2al \cos a + y^2)^{3/2}} da dy \quad (2)$$

in which the limits of y are $(-h, +h)$, and the limits of a are $(-\cos^{-1} \frac{a}{l}, +\cos^{-1} \frac{a}{l})$.

The integral¹ of the above expression is

$$J = \frac{i}{l} \left\{ a \cos^{-1} \frac{l^2 - a^2 - h^2}{l^2 - a^2 + h^2} + h \left[(q+1) \sqrt{\frac{1}{q}} \cot^{-1} \sqrt{\frac{p}{q}} - 2 \cot^{-1} \sqrt{p} \right] \right\} \quad (3)$$

where

$$p = \frac{l+a}{l-a}, \quad q = \frac{(l+a)^2 + h^2}{(l-a)^2 + h^2} \quad (4)$$

¹ This integral was obtained by Prof. A. S. Chessin, of Washington University, to whom the writer desires to express his indebtedness.

It only remains to substitute for the constants a and h in the above equation the numerical values pertaining to a Nernst glower and then to compute the value of J for different distances, l . Before doing this, however, it is interesting to note, in passing, the form which the equation assumes when we make h approach infinity. Under this condition g approaches unity, and so in the limit, equation (3) becomes

$$J = \frac{\pi i a}{l}$$

In other words, the intensity of illumination due to an infinitely long uniformly radiating cylinder varies inversely as the first power of the distance from the axis of the cylinder, a result which also follows from purely physical considerations of the normal flow of energy across coaxial cylindrical surfaces.

Let us now substitute the dimensions of the Nernst glower in equation (3), compute the intensity of illumination at different distances, l , and compare the relative values with those obtained on the assumption of the inverse square law. Two sizes of glowers were used in the investigation, 88-watt glowers and 44-watt glowers. The dimensions of the former are $h=7.5$ mm, $a=0.6$ mm. Of the latter, $h=6$ mm, $a=0.4$ mm. If, then, in the above equation we put $h=10$ mm and $a=1$ mm we shall include both glowers, and

TABLE I.

Deviation from the Inverse Square Law of the Radiation of a Cylinder 20 mm long and 1 mm radius.

Distances, l	J (Direct Evaluation)		J (Inverse Square Law)		Deviation
3000	1.0000	$\times J_{3000}$	1.0000	$\times J_{3000}$	$\pm 0.00 \%$
2000	2.2500	"	2.2500	"	+0.00 "
1000	9.0045	"	9.0000	"	+0.05 "
500	3.6040×10	"	3.6000×10	"	+0.11 "
200	2.2545×10^2	"	2.2500×10^2	"	+0.20 "
100	9.0081×10^2	"	9.0000×10^2	"	+0.09 "
80	1.4049×10^3	"	1.4062×10^3	"	-0.09 "
50	3.5593×10^3	"	3.6000×10^3	"	-1.13 "

the deviations from the inverse square law which we shall find will be greater than those incident to the use of either glower. The

results of this substitution for different distances, l , are shown in Table I and Fig. 3. In the first column of Table I are given the distances, l , expressed in millimeters, for which the values of J were computed. The second column contains the ratios of the intensities at the different distances, l , to that at $l=3000$ mm, as obtained by direct substitution in equation (3); and the third column gives the same ratios obtained by the inverse square law. The percentage errors are shown in the fourth column, in which (+) means that the intensity of illumination determined by direct evaluation of the integral of equation (3) is relatively greater, as compared with the intensity at $l=3000$ mm, than the value obtained by the inverse square law from the intensity at $l=3000$ mm. This is shown in Fig. 3, in which abscissas are distances, l , and ordinates are per-

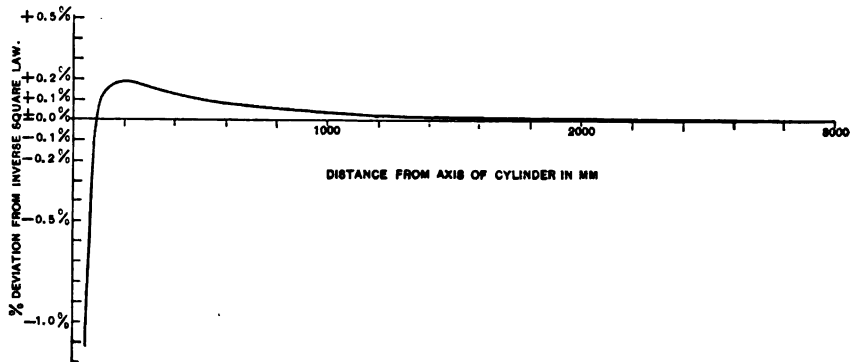


Fig. 3.—Deviation of Radiation of Cylinder from Inverse Square Law.

centage deviations from the inverse square law, as deduced from the value of the intensity at $l=3000$ mm. It is thus seen that the value of J between $l=3000$ mm and $l=90$ mm is greater (as compared with J at $l=3000$ mm) than the value of J deduced on the assumption of the inverse square law, the maximum deviation being about +0.2 per cent. At distances less than about 90 mm the intensity becomes very much less than the values demanded by the inverse square law, the deviation at $l=50$ mm being over 1 per cent.

Since in the experiments described below the extreme distances used were $l=3000$ mm and $l=500$ mm the maximum error on the assumption of the inverse square law for the Nernst is approximately 0.1 per cent, which is entirely negligible. Moreover, the errors given in Table I and Fig. 3 were deduced for a cylinder having the

dimensions $h = 10$ mm, $a = 1$ mm, which are somewhat greater than the dimensions of an 88-watt Nernst glower, and much greater than the dimensions of a 44-watt glower. Hence the true errors for the Nernsts are probably less than those given in the table.

It should be stated that the above theory was based on the assumption of a true circular cylinder with constant specific light intensity, i . The Nernst glower does not fulfill either of these conditions rigorously. It frequently becomes warped after burning for some hours, and direct experiment reveals the fact that the intensity, i , is different at different parts of the filament. These deviations probably remain constant, however, during a series of experiments, and so only introduce differential errors in the relative intensities of illumination at different distances.

(b) *Method of Mean Distances.*—In the above investigation l was defined to be the distance from the center of the cylinder to the photometer screen, the axis of the cylinder being parallel to the screen, and the centers of both cylinder and screen lying in the same plane perpendicular to the axis of the cylinder. In the experimental method employed it was found difficult to measure this distance directly by the use of the scale on the photometer bar. The glower was therefore mounted in a horizontal position in a socket that could be turned about a vertical axis through its center. The position of the axis of rotation on the bar was shown by an index attached to the carriage on which the socket was mounted, so that distances between the axis of rotation and the photometer could be read off the scale on the bar. The center of the glower was now placed approximately in the axis of rotation, and readings were made first with one side of the glower, [designated (+)] and then with the other side of the glower [designated (—)] turned toward the photometer. In each case the distance was measured to the axis of rotation and the mean of the two readings taken. This mean distance was the distance used in the comparison of intensities of illumination, and it must hence be shown that the square of the ratio of one mean distance, x , to another mean distance, x' , gives the true value of the ratio of the two corresponding illuminations, J and J' .

Before proceeding to this let us consider a different way of interpreting the deviations of the law of radiation of the cylinder from the inverse square law. Since for all distances greater than $l = 90$

mm the intensity of illumination compared with the intensity at $l=3000$ mm is slightly greater than that for a point source, we may consider the cylinder as a point source, the effective center of radiation of which is at the center of the cylinder for $l=3000$ mm, but

for distances between $l=3000$ mm and $l=90$ mm it is displaced a variable small amount toward the photometer screen. Hence, as the deviations from the inverse square law are very small for all distances used in the experiments, we can take the percentage error in distance to be one-half the percentage error in intensity of illumination. If, therefore, we take one-half the ordinates of Fig. 3 we obtain the percentage errors in the effective distances. In this way we can determine for any distance the displacement, c , of the effective center of radiation from the geometrical center of the cylinder.

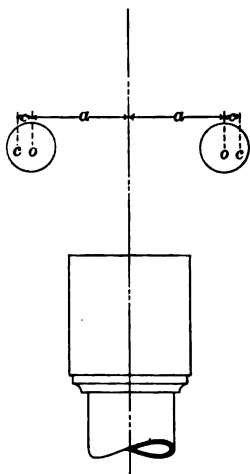


Fig. 4.—Displacement of Axis of Cylinder from Axis of Socket.

We will suppose now that the geometrical center of the cylinder is at a distance, a , from the axis of the socket (Fig. 4). Suppose further that the photometer screen has an intensity of illumination, J , the distance from the axis of the socket to the screen being x_1 when the (+) side of the Nernst is turned toward the screen, and x_2 when the (−) side is turned in that direction. If now I_1 and I_2 are the effective intensities of the (+) and (−) sides of the Nernst, respectively, considered as a point source,

$$J = \frac{I_1}{(x_1 - a - c)^2} = \frac{I_2}{(x_2 + a - c)^2} \quad (5)$$

In a similar way the equation corresponding to an illumination, J' is

$$J' = \frac{I_1}{(x_1' - a - c')^2} = \frac{I_2}{(x_2' + a - c')^2} \quad (6)$$

Hence the true ratio of J to J' is

$$\frac{J}{J'} = \left(\frac{x_1' - a - c'}{x_1 - a - c} \right)^2 = \left(\frac{x_2' + a - c'}{x_2 + a - c} \right)^2 \quad (7)$$

which, by composition, becomes

$$\frac{J}{J'} = \left(\frac{x_1' + x_2' - 2c'}{x_1 + x_2 - 2c} \right)^2 \quad (8)$$

Or, if we put

$$x = \frac{x_1 + x_2}{2}, \quad x' = \frac{x_1' + x_2'}{2} \quad (9)$$

$$\frac{J}{J'} = \left(\frac{x' - c'}{x - c} \right)^2 \quad (10)$$

$$\begin{aligned} &= (x'^2 - 2c'x' + c'^2) \left(\frac{1}{x^2} + \frac{2c}{x^3} + \frac{3c^2}{x^4} + \text{etc.} \right) \\ &= \left(\frac{x'}{x} \right)^2 \left(1 - \frac{2c'}{x'} + \frac{c'^2}{x'^2} \right) \left(1 + \frac{2c}{x} + \frac{3c^2}{x^2} + \text{etc.} \right) \\ &= \left(\frac{x'}{x} \right)^2 \left(1 - \frac{2c'}{x'} + \frac{2c}{x} \right) \end{aligned} \quad (11)$$

to within negligibly small quantities. From this equation, or better from equation (10), it follows that for a true point source at the center of the cylinder the ratio of the intensities obtained by taking the mean distances x and x' is the same as that obtained from the actual distances $x_1 - a$ and $x_1' - a$, or $x_2 + a$ and $x_2' + a$, since for a true point source $c = c' = 0$ and from equations (7) and (10)

$$\frac{J}{J'} = \left(\frac{x_1' - a}{x_1 - a} \right)^2 = \left(\frac{x_2' + a}{x_2 + a} \right)^2 = \left(\frac{x'}{x} \right)^2 \quad (12)$$

In the actual case of the cylinder, c and c' have very small values, as pointed out above. In fact since $\frac{c}{x} \times 100$ is the percentage distance correction, and $\frac{2c}{x} \times 100$ is the percentage deviation of the intensity from the inverse square law, it follows that $\frac{2c}{x}$ is $\frac{1}{100}$ of the ordinate of the curve of Fig. 3 expressed as a decimal fraction, or is equal to the ordinate of the curve expressed in per cent. Since the greatest range of distance used was between $l' = [x' - a] = 3000$ mm

and $l = [x - a] = 500$ mm we see from the curve of Fig. 3 that $c' = 0$ and $c = +0.11$ per cent, so that [eq. (11)]

$$\frac{J}{J'} = \left(\frac{x'}{x}\right)^2 (1 - 0 + 0.11 \%) \quad (13)$$

in which the correction is negligible.

Hence the method employed of taking the mean of the (+) and (−) readings gives absolutely true results for a point source, and for a cylinder the errors are only those incident to the deviation from the inverse-square law of the radiation of the cylinder.

3. APPARATUS AND METHODS.

(a) *General Method.*—The photometer bench on which the measurements were made is a Reichsanstalt precision photometer bench 250 cm long, supplied with a Lummer-Brodhun contrast screen. In the experiments with disks of small opening it was necessary to use greater distances than could be obtained on the 250 cm bar. In these cases an extension was fitted to one end of the bench, increasing its length by about 175 cm. In all the experiments except those in which there was a decided color difference the contrast principle was used but in the cases of great color difference the absorption strips producing the contrast were removed and the setting was made for a match.

Throughout the experiments the substitution method was employed. Thus the source at the right end of the bar was merely used as a comparison lamp.¹ The carriage on which it was mounted was connected rigidly by means of adjustable links to the carriage supporting the photometer screen and at a suitable distance to produce the desired illumination of the screen. The comparison lamp thus moved with the screen, remaining at a constant distance from it, except in so far as slight irregularities in the ways produced small variations in the distance, depending upon the positions of the carriages on the bar. These eccentricities were carefully studied for all positions of both carriages on the bar and the results were tabulated, so that the variations in the distance between the comparison lamp and the screen, as the latter was moved back and forth

¹ Hereafter the lamp on the right will be referred to as the "comparison lamp," in distinction from the "test lamp" on the left.

on the bar, could be determined directly. Since, however, in nearly every experiment the position of the photometer was approximately the same with the sectored disk stationary and rotating, and since regions of the bar were selected which were approximately true, in general no correction was necessary for the distance between the comparison lamp and the screen. Hence the illumination on the right side of the screen remained constant to within the variations of the comparison source itself.

On the left side of the photometer the test Nernst and the sectored disk were mounted. Fig. 5 is a diagrammatic sketch of the

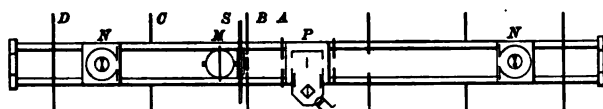


Fig. 5.—Diagrammatic Sketch of Photometer Bench.

general arrangement, in which A, B, C, and D are diaphragms covered with black velvet, which were always so arranged that no stray light could reach the photometer screen except by reflection from the black velvet at an angle approximately equal to 90° . The magnitude of this stray light, when the walls of the room were white and were rather strongly illuminated, was found by a previous investigation¹ to be entirely negligible. *M* is a small spherical motor, which carries on its shaft the sectored disk *S*. *NN* are the two sources and *P* is the photometer screen.

(b) *Sources*.—Direct current Nernst glowers were used throughout the investigation for the test source at the disk end of the bar, and in most of the measurements a glower was used for the comparison lamp. When the effect of color difference was investigated an incandescent lamp at low voltage was used for a comparison lamp, and in some few experiments on colored light an incandescent lamp was substituted for the comparison Nernst. As stated above, the glowers were burned without any globes around them, since reflection, refraction, and diffusion of the light by the globe might produce serious errors. Moreover the

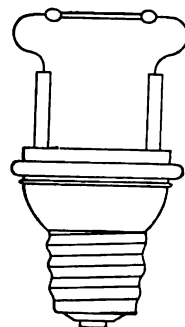


Fig. 6.—Nernst Glower Mounted on Extension Plug.

¹ Bureau of Standards Bulletin No. 3, p. 417.

glowers were not provided with the customary ballast resistance. They were mounted on extension plugs, as shown in Fig. 6, and were operated with constant current on a storage-battery circuit of 250 volts, of which approximately 150 volts was taken up in rheostats. This series resistance took the place of the customary ballast resistance and served to keep the current approximately constant.

The electrical measurements were made in terms of a Weston standard cell by means of a potentiometer. Thus, the current was measured by the difference in potential across a standard resistance and the voltage was obtained by the use of a 100,000 ohm multiplier.

The original plan was to leave the Nernst glowers entirely uncovered, but it was found that due to air currents the resistance of the glowers changed continually, so that it was impossible to maintain the current in the glowers constant, and the fluctuations in the volt-

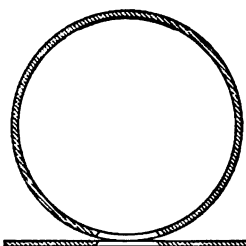


Fig. 7.—Horizontal Section of Hood through Opening.

age were so great as to preclude the possibility of measurement closer than to several tenths of 1 per cent. Moreover, the effect of the rotating disk in the neighborhood of the Nernst was evident both in the current and in the voltage. I was thus led to try the effect of partially surrounding the Nernsts with hoods. To this end two pieces of brass tubing 40 cm long and 15 cm diameter were each provided with a rectangular opening 1 cm high and 5 cm wide at a distance of 28 cm from one end of the tube, so as to be opposite the Nernst when placed on the base of the carriage on which the Nernst was mounted. Each of these hoods was provided with a loosely fitting piece of sheet brass for a cover and was painted a dead black. Lest any stray light might be reflected into the photometer at approximately grazing incidence from the outside of the tube, a sheet of brass 25 cm long and 15 cm wide was covered with black velvet and fastened to the front of the hood with the black velvet turned toward the screen. This diaphragm was provided with a rectangular opening to correspond with the opening in the hood, as shown in Fig. 7, which is a horizontal section in the plane of the opening. That part of the inside of the tube which was behind the Nernst was also covered with black velvet in order to prevent any possible reflection from the painted metal surface.

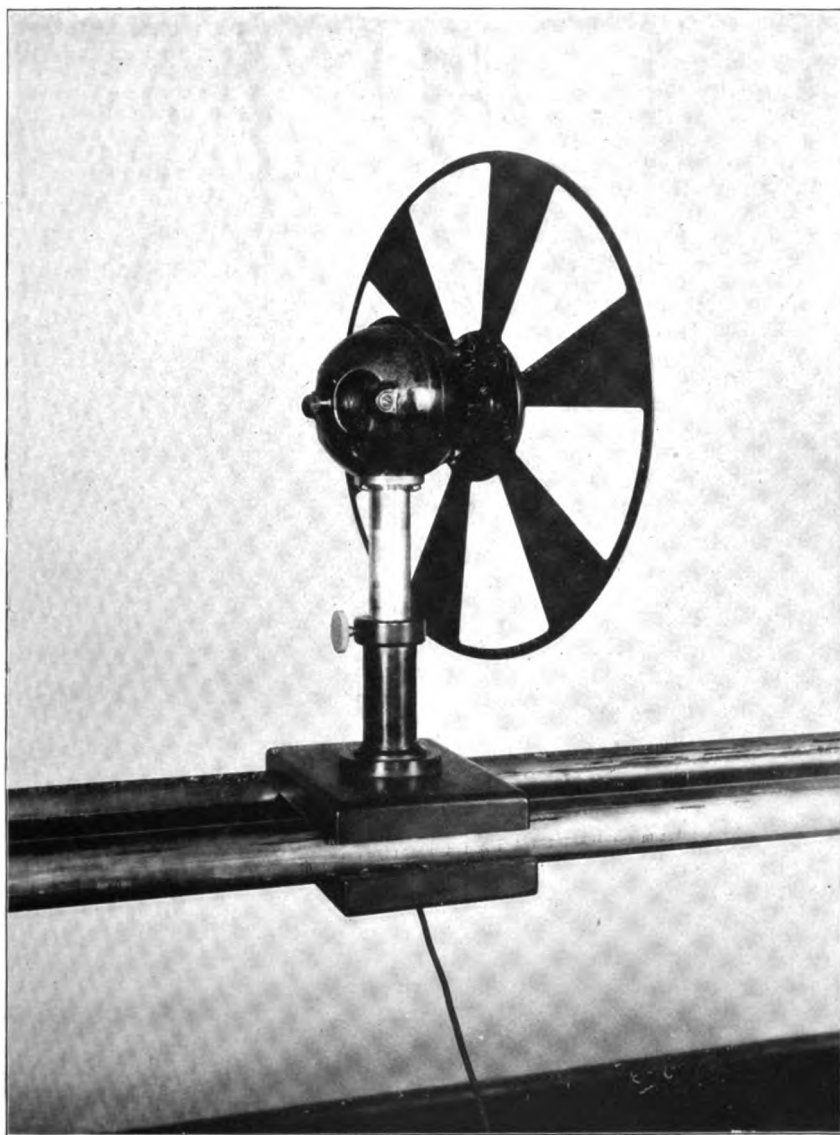


Fig. 8.—*Sectored Disk Mounted on Photometer Bar.*



Under these hoods the Nernsts operated almost as steadily as incandescent lamps. The current could be maintained constant to within 1 or 2 parts in 10,000, and the voltage could usually be measured to the same accuracy. Moreover, the luminous intensity remained constant to within the limit of observational error, and the sector-disk in the neighborhood of the Nernst seemed to have no appreciable effect on its voltage at constant current, which affords a quite sensitive means of determining slight variation in the temperature of the Nernst. It was for this reason that voltage measurements were made with every series of observations.

One peculiar voltage effect was noticed which occurred at times both when the disk was stationary and when it was rotating. The voltage would suddenly change over several tenths of a volt and then gradually creep back. As this change was usually accompanied by no perceptible variation in the luminous intensity, if the current was maintained constant, and as it seemed to occur more frequently after the Nernsts had burned for many hours, it seemed probable that it was due to the cracking of the terminals of the filament and a consequent redistribution of the current at the terminals. This hypothesis seemed to be borne out by the fact that in all old Nernsts the positive ends show large and deep cracks.

At first the presence of these sudden changes in the voltage caused doubt as to the accuracy of the results. They repeatedly occurred both while the disk was stationary and while rotating, and could not be detected at the photometer, although every other condition was maintained constant. I was therefore led to disregard them, particularly as the results obtained in sets in which there was no trace of such an effect agreed with those obtained in sets in which the effect was plainly noticeable.

Having shown that the Nernst could be depended upon for its constancy, it remained to find some convenient way of determining the distance between the center of the test Nernst and the photometer screen. The test Nernst (as distinguished from the comparison Nernst) mounted on an extension plug, was screwed into the socket of a horizontal rotator with the driving shaft removed. But although the distance from the axis of the rotator to the near surface of the plaster of Paris screen in the photometer-head was

known for every position of the rotator and of the photometer-head, as the result of a direct calibration of the scale on the bar, the distance of the center of the Nernst from the axis of the rotator varied slightly from time to time, due to bending of the terminal wires of the Nernst on heating, etc. Hence, after each series of measurements it was necessary to re-determine the distance. It therefore occurred to the writer to use the method described briefly in the above theory. A double pointer was attached to the rotator, and by means of a corresponding pointer attached to the carriage the Nernst could be turned through 180° . The mean of the (+) and (−) readings as described above was taken in each case.

(c) *Disks.*—The rotating sector disk is shown in Fig. 8. The rotating device consisted of a small spherical direct current fan motor, 9 cm in diameter, mounted on a base clamped to the ways. In the figure the motor and disk are turned slightly with reference to the base, so as to be seen better. The distance of the motor above the ways could be adjusted by means of the telescoping tubes which formed the column on which the motor was mounted. The frame of the motor was japped. In order to prevent light from being reflected into the photometer-head from the smooth surface of the motor, the top of the motor was at first covered with black velvet, but since the light was incident on this at nearly grazing incidence it was thought better to substitute a small metal screen placed normal to the direction of the rays. This was accomplished by separating the two halves of the spherical shell of the motor and inserting a blackened piece of thin sheet aluminum cut to fit. This is shown in the figure.

The sector disks were mounted directly on the shaft of the motor and the height of the motor was adjusted so that the sector cut the beam of light normally, the central ray of the beam lying in the same vertical plane as the shaft of the motor. Instead of the customary method of using a graduated variable sector disk, it was decided to use separate disks for the different angular openings. This was decided upon partly because four such disks were at hand and partly because there was less chance for error in the calibration of the opening. The original four disks were made of aluminum about 1.5 mm thick, and each contained six open and six solid sectors equally spaced. The total angular openings (which will here-

after always be meant when reference is made to the opening of a disk unless it is stated otherwise) were 288° ($6 \times 48^\circ$), 270° ($6 \times 45^\circ$), 240° ($6 \times 40^\circ$), and 180° ($6 \times 30^\circ$). Subsequently the following disks were added: One steel disk, 1.5 mm thick, with total opening of 60° in three sectors of 20° each; two hard brass disks, 1.5 mm thick, with total openings of 210° in six sectors of 35° each, and 30° in six sectors of 5° each.

All of the disks were approximately 30 cm in diameter, and had the same general form as that shown in Fig. 8. They were all painted dead black, and the last two disks (210° and 30°) had the edges of the sectors beveled so as to prevent reflection from them.

With these seven disks the law could be tested at seven points, ranging irregularly from 30° to 288° . In order to obtain more points of the curve it was decided to cover some of the openings of the disks with thin pieces of blackened aluminum clamped over the openings (Fig. 9). The covers were always placed symmetrically, so that with a disk of six open sectors it was possible to obtain either one-half the total opening by covering the alternate sectors, or to obtain one-third the total opening by covering all but two opposite sectors. Moreover, by changing the covers it was possible to obtain the same opening in different ways, i. e., by using different sectors of the same disk.

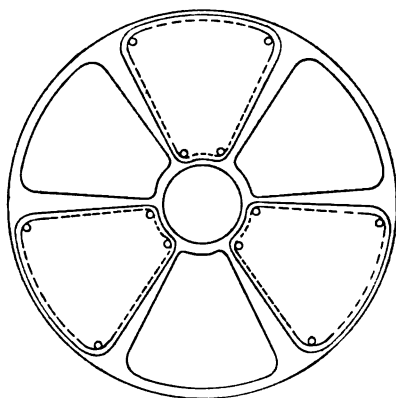


Fig. 9.—Sector Disk with Three Openings Covered.

By this method the following total openings were obtained: 10° , 15° , 30° , from the 30° disk; 60° from the 60° disk, and also from the 180° disk; 90° from the 180° disk and also from the 270° disk; 120° from the 240° disk; 144° from the 288° disk; 180° , 210° , 240° , 270° , and 288° from the respective disks.

The disks were all carefully calibrated on a circular dividing engine. The opening of each sector of each disk was measured at seven successive radial distances 0.5 cm apart, over a range of 3 cm, since 3 cm was approximately the maximum height of the beam

at the point where it was intercepted by the disk. The mean of the seven readings was taken as the opening of the sector and the height of the disk was always so adjusted that the line from the center of the Nernst to the center of the photometer screen intercepted the disk at the middle point of the region over which it had been calibrated. The average deviation for different radial distances from the mean value for any combination of sectors that was ever used was always under 0.1 per cent, but owing to difficulties in setting on the edges, the total opening was perhaps not known to an accuracy much better than 0.1 per cent. The values of the different sectors of the various disks are given in the "Experimental Results."

The sectored disk in all cases, with one or two exceptions which will be noted later, was placed between diaphragms *B* and *C* (Fig. 5) and as close to *B* as possible. This diaphragm had a circular aperture of 10 cm and hence screened the motor (except the upper part) and its support from the photometer screen. The distance between the disk and the photometer screen ranged from 20 to 25 cm. Moreover, the sectored disk was always left in position with the open sector in the path of the light for readings intended to give the direct intensity, except when the 30° disk was used. The 5° sectors of this disk were too narrow to permit the beam to pass. Measurements made with the larger disks removed, however, failed to show any effect of the stationary disks.

The complete arrangement, except the extension ways, is shown in Fig. 10 which was made from a photograph. The hood was removed from the comparison Nernst and is shown standing on the table.

(d) *Details of Methods—White Light.*—The method of conducting the experiments was as follows: Two seasoned Nernst glowers were mounted in the two sockets and the test Nernst was adjusted so that its center was at the proper height, coinciding very nearly with the axis of the rotator, and the direction of its axis was perpendicular to the line joining its center to that of the photometer screen. The two Nernsts were then brought to incandescence and allowed to burn about an hour. The test Nernst was placed at some definite position and the connecting links between the photometer screen and the comparison Nernst were adjusted until the balance came at

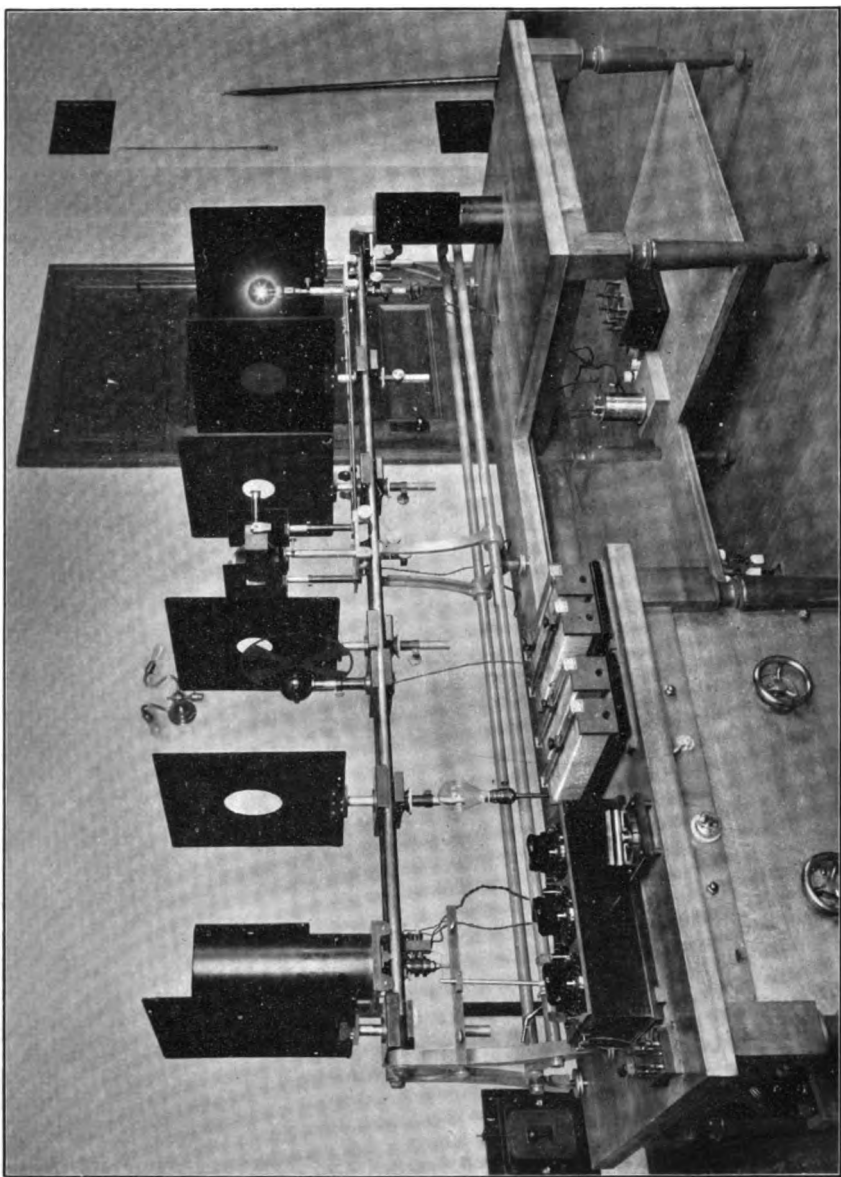


Fig. 10.—Complete Photometer Bench.



a desirable part of the bar. While one observer maintained the currents constant the other observer made the photometric settings. First, with the disk stationary, a series of three or more readings was made with the Nernst in the (+) position, then a second series with the Nernst in the (−) position, and then two more series, one (+) and the other (−), making in all four series of measurements with the disk stationary. The sector disk was then set rotating and the test Nernst was moved in nearer to the photometer screen to some position such that the balance came at about the same position on the photometer bar as before. The carriage holding the test Nernst was then clamped to the ways, and kept at this fixed position while the settings for a balance were made. As explained before, the settings were made by moving the photometer screen and comparison Nernst, which moved as a single system remaining at a constant fixed distance apart during the entire experiment. Four series of three readings each were then made as with the disk stationary. This process of alternate stationary and rotating measurements was continued until three groups of stationary readings; and two groups of rotating readings had been made, making a total of twelve series of stationary readings, six (+) and six (−), and eight series of rotating readings, four (+) and four (−). The square of the ratio of the mean distance between the photometer screen and the axis of the rotator, when the disk was rotating, to the mean distance when the disk was stationary, was taken as the effective ratio of the light transmitted by the disk to that incident on it. The comparison of this ratio with the ratio of the angular opening of the disk to 360° gave the deviation from the law for the disk used.

This method was used for all the disks except in some few sets near the end of the investigation in which only three groups of readings instead of five were made. In all cases after each series of three readings the voltages on the two Nernsts were measured and recorded.

Colored Light.—In the measurements with colored light the method was exactly the same as that used for white light except that a piece of colored glass was introduced into the eyepiece of the photometer. Red, green, and blue glasses were used and measurements were made on three disks, the 240° disk, the 60° disk,

and the 15° disk. The colored glasses used were not monochromatic, but since to the eye the light appeared distinctly red, or green, or blue, as the case may be, any appreciable error due to one of the colors would probably not be modified greatly by the admixture of relatively small quantities of light of another color.

4. EXPERIMENTAL RESULTS.¹

(a) *Effect of Speed.*—Before beginning the investigation of Talbot's law, the effect of the speed of rotation of the disk on the illumination of the screen was tested. Since numerous investigators have found that the speed of the disk has no effect on the apparent illumination of the screen provided the speed is greater than a critical speed below which fluctuations in the intensity are visible, it seemed unnecessary to make an extended study of this effect. It was desirable, however, to check the conclusion of previous investigators for a range of speed likely to be used in the experiments. Hence, using successively two disks—the 180° disk and the 60° disk—the speed of the disk was varied from the "flicker" point to the maximum speed attainable and no variation of the illumination within the range of experimental error was detected. Subsequently, therefore, no attention was paid to the speed of the disk except to be sure that the speed was sufficiently high, i. e., that the number of alternations of the sectors was sufficiently great to prevent the possibility of a "flicker."

(b) *Disks.*—In order to show the accuracy of the mechanical construction of the sectorized disks, the readings on the 30° disk for the six separate sectors and at different distances from the periphery of the disk are given in Table II. The first column contains the distances, expressed in centimeters, from the periphery of the disk to the point at which the measurements were made. The other six columns contain the readings on the angular openings of the six sectors numbered from I to VI.

Since the 30° disk contained the smallest sectors of all the disks it had to be made and measured most carefully of all, for small deviations of the edges of the sectors from straight radial lines would produce relatively large errors in the angular openings. The absolute angle need not be made to agree very closely with the nominal value, as the subsequent calibration will give the true value of

¹ I wish to express my indebtedness to my assistants, Mr. F. E. Cady and Mr. A. H. Schaaf, for valuable assistance both in observations and computations.

the opening, but it is very necessary that the edges be made approximately radial and quite straight, so that the mean of the seven measurements made every 5 mm may represent the mean value of the opening. It is seen from Table II that the maximum range of variation at different radial distances for most of the sectors is only

TABLE II.

Calibration of 30° Disk.

Distance from Periphery	I	II	III	IV	V	VI
cm						
3.0	5°04'48''	5°03'46''	5°03'02''	5°04'45''	5°05'28''	5°06'24''
3.5	05 08	03 54	03 00	04 28	05 27	06 22
4.0	05 17	03 50	03 12	04 26	05 23	06 40
4.5	05 21	03 42	03 00	04 20	05 14	06 34
5.0	05 06	03 52	02 54	04 14	05 28	06 57
5.5	05 10	03 51	02 45	04 14	05 33	07 10
6.0	05 14	03 51	02 54	04 28	05 44	07 08
Mean	5 05 09	5 03 49	5 02 58	5 04 25	5 05 28	5 06 45

about 30'' or 1 part in 600. The average deviation from the mean opening would be well under 1 part in 1,000.

As stated above, the disks were always mounted in such a position that the beam of light passed through the calibrated part of the sectors.

Measurements similar to those for the 30° disk were made on all the disks, the final results of which in the form in which they were used are given in Table III. The first column contains the various disks. The second column contains the nominal angles obtained from the different disks, and the third column gives the sectors that were employed to obtain these angles. In the fourth column are given the ratios of the actual angles to 360°. These ratios, compared with the ratios of the squares of the distances obtained in the experiments, gave the errors in the law.

TABLE III.

Disks	Nominal Angles	Sectors Used	Corresponding Angles +360°
30°	30°	I-VI	0.08466
	15°	I, III, V	0.04229
	15°	II, IV, VI	0.04236
	10°	I, IV	0.02822
	10°	II, V	0.02821
	10°	III, VI	0.02823
60°	60°	I-III	0.16914
180°	180°	I-VI	0.5012
	90°	I, III, V	0.2506
	90°	II, IV, VI	0.2507
	60°	I, IV	0.16697
	60°	II, V	0.16718
	60°	III, VI	0.16710
210°	210°	I-VI	0.5841
240°	240°	I-VI	0.6683
	120°	I, III, V	0.3341
	120°	II, IV, VI	0.3342
270°	270°	I-VI	0.7508
	90°	I, IV	0.2501
	90°	II, V	0.2504
	90°	III, VI	0.2503
288°	288°	I-VI	0.8000
	144°	I, III, V	0.4000
	144°	II, IV, VI	0.4000

(c) *White Light*.—In order that the exact method of observation and computation may be known, the following series of measurements of June 2, 1905, copied from the laboratory record book, is given:

SERIES XXV.

60° disk, total opening 60.89° $\frac{60.89}{360} = 0.16914$. 88-watt Nernst.
No. 9 on left at 0 when disk is stationary, and at 71.5 when disk is

rotating. 88-watt Nernst No. 10 on right, as comparison lamp, at fixed distance 119.9 cm from photometer screen.

60° disk stationary.

(+)		Volt.
122.97		No. 10=111.01
.82	122.78	No. 9=117.08
.36		
.75		

(-)		Volt.
122.33		No. 10=110.98
.07	122.19	No. 9=117.08
.17		

Nernst No. 9 at 0.

(+)		Volt.
122.97		No. 10=110.99
.63	122.80	No. 9=117.08
.80		

(-)		Volt.
122.03		No. 10=111.00
1.98	121.99	No. 9=117.07
1.97		

60° disk rotating.

(-)		Volt.
121.91		No. 10=111.01
.89	121.87	No. 9=117.07
.80		

(+)		Volt.
122.44		No. 10=111.00
.22	122.26	No. 9=117.07
.23		
.17		

Nernst No. 9 at 71.5.

(-)		Volt.
121.91		No. 10=111.00
.78	121.84	No. 9=117.07
.82		

(+)		Volt.
122.38		No. 10=110.99
.16	122.24	No. 9=117.04
.17		

60° disk stationary.

(+)		Volt.
122.61		No. 10=111.00
.67	122.69	No. 9=117.03
.80		

(-)		Volt.
122.33		No. 10=110.98
2.10	122.11	No. 9=117.02
1.94		
2.07		

Nernst No. 9 at 0.

(+)		Volt.
122.97		No. 10=110.99
.71	122.85	No. 9=117.06
.99		
.73		

(-)		Volt.
122.02		No. 10=110.97
.13	122.06	No. 9=117.03
.02		

60° disk rotating.

(-)		Volt.
121.77		No. 10=110.99
.87	121.82	No. 9=117.04
.82		

(+)		Volt.
122.31		No. 10=110.99
.37	122.30	No. 9=117.04
.22		

Nernst No. 9 at 71.5.

(-)		Volt.
121.89		No. 10=110.99
.87	121.85	No. 9=117.04
.80		

(+)		Volt.
122.20		No. 10=110.98
.31	122.24	No. 9=117.05
.21		

60° disk stationary.

(+)		Volt.
122.73		No. 10=110.98
.81	122.81	No. 9=117.02
.97		
.72		

Nernst No. 9 at 0.

(+)		Volt.
122.83		No. 10=111.00
.64	122.68	No. 9=117.01
.57		

60° disk stationary.			Nernst No. 9 at 0.		
(-)		Volt.	(-)		Volt.
122.01		No. 10=110.99	121.97		No. 10=111.01
2.29		No. 9=117.02	2.01		No. 9=117.02
1.88	121.99		1.77	121.92	
1.87			1.93		
1.89					

COMPUTATIONS.

Disk stationary.			Disk rotating.		
(+)	122.78 .80	122.79	(+)	122.26 .24	122.25
		122.44			122.06
(-)	122.19 1.99	122.09	(-)	121.87 .84	121.86
(+)	122.69 .85	122.77	(+)	122.30 .24	122.27
		122.42			122.06
(-)	122.11 .06	122.08	(-)	121.82 .85	121.84
(+)	122.81 .68	122.74			
		122.35			
(-)	121.99 .92	121.96			

Mean stationary setting	122.40	Mean rotating setting	122.06
Position of Nernst carriage	0.00	Position of Nernst carriage	71.50
Distance=	122.40	Distance=	50.56
Correction to bar	-0.41	Correction to bar	-0.32
Corrected distance=	121.99	Corrected distance=	50.24

$$\left(\frac{50.24}{121.99}\right)^2 = 0.16961.$$

$$\frac{60.89^\circ}{360^\circ} = \frac{0.16914}{+47} = +0.28\%. \quad \text{Deviation from Talbot's law.}$$

This method of observation and computation was followed throughout the investigation. A great number of different Nernst glowers, both of the 88-watt type and of the 44-watt type, were used. The choice between an 88-watt glower and a 44-watt glower for the comparison lamp was determined by convenience. For the test lamp, however, measurements were made with both types with the larger angular openings in order to test the effect of change in the absolute illumination of the screen. Thus with the 288°, 270°, 240°, and 180° disks measurements were made with both the 44-

watt and the 88-watt glowers, each at two different distances, so that the extreme illuminations used were in the ratio of 1:4. With the 60° disk illuminations in the ratio of 1:2 were used, but in no case was there any evidence of an effect due to absolute intensity of illumination. These variations in the intensity of illumination were not very large, so that it would be interesting to study the effect of large changes in the intensity of illumination, particularly for disks with small openings. In my observations with openings smaller than 60° 88-watt glowers were always used, since even with them the illumination of the photometer screen was quite small, because of the necessarily long distance. Thus with the 10° opening the distance between the test Nernst and the screen was approximately 300 cm when the disk was unmounted and only 50 cm when the disk was rotating. Moreover, because of this great range of distance but one distance for each position could be used conveniently.

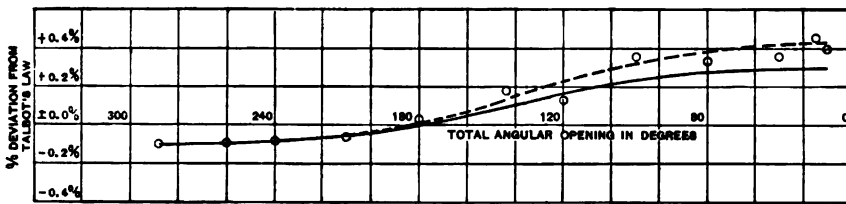


Fig. 11.—Deviation from Talbot's Law.

The results obtained with the various angular openings are shown in Table IV, in which (+) means that the disk apparently let through more light than would be expected from Talbot's law. It should be stated that the order in which the disks were used was most irregular. Several of them, for example, were tested both near the beginning and near the end of the investigation, which extended over a period of ten or twelve months.

The mean values for the different angular openings are plotted in the form of a curve in Fig. 11, in which abscissas are angular openings, and ordinates are percentage deviations from the law of Talbot. The dotted curve represents the values given in Table IV.

It is seen that for all angular openings from 288° to 10° the law is verified to within 0.5 per cent. Attention should be called, however, to the form of the curve. Since the average deviation of the readings for any one angle was in no case as large as 0.2 per cent,

the probable errors of the measurements are in all cases under 0.1 per cent, while the observed errors in the law are as large as 0.4 per cent. Because the probable errors are less than 0.1 per cent, it does not necessarily follow, however, that the results are correct to the

TABLE IV.

Percentage Deviations from Talbot's Law Using White Light.

Total Angular Openings	120°	90°	60°	30°	15°	10°
	-0.16%	-0.05%	-0.05%	-0.06%	+0.10%	+0.24%
	+ .01	.03	- .12	- .05	+ .14	.25
	- .09	.06	- .17		- .15	.09
	- .12	.16	+ .01		- .10	
					+ .13	
					+ .11	
Means	-0.09	-0.08	-0.08	-0.06	+0.04	+0.19
Average variation from the mean ..	±0.05	± .04	± .06	± .00	± .11	± .07

Table IV, continued.

Total Angular Openings	120°	90°	60°	30°	15°	10°
	+0.22%	+0.37%	+0.54%	+0.35%	+0.38%	+0.14%
	.04	.51	.15	.37	.63	.52
		.39	.20	.39	.52	.38
		.33	.20		.55	.61
		.40	.20		.61	.39
		.45	.28		.26	
		.03	.65		.26	
		.41	.44		.40	
Means	+0.13	+0.36	+0.33	+0.37	+0.45	+0.41
Average variation from the mean ..	± .09	± .09	± .16	± .01	± .13	± .13

same accuracy. Some constant source of error might possibly be present. It will be noticed from the table that approximately the same deviation was found for all the openings less than 120°. Now, in all the measurements on these angles, the rotating readings were

taken with the test Nernst about 50 cm from the photometer screen, at which distance the entire system was very much congested. Hence, it seemed possible that some stray light might have penetrated to the screen which would not have reached the screen when a longer distance was used as in the stationary readings. In particular the photometer screen itself reflects light back on the diaphragms and on the solid sectors of the disk, and since these were necessarily very close to the screen when the test Nernst was only 50 cm away, the apparent deviation from the law might possibly be due in part to this stray light.

An attempt was made to measure the effect of this stray light by cutting off the direct rays from the Nernst, but it was found very difficult to reproduce the conditions, and so this method of attack was abandoned. A second method which seemed productive of results, but which could not be carried as far as desired for want of time, consisted in using longer distances for the 90° disk and in comparing the 15° disk with the 180° disk, which has an error of $+0.04$ per cent. Only a few observations were made, but they seemed to be in the right direction. Thus, with the 90° disk four determinations at a much greater distance than that used before gave as the errors $+0.38$ per cent, $+0.14$ per cent, 0.09 per cent, and $+0.18$ per cent, with a mean of $+0.20$ per cent, as against $+0.36$ per cent in the previous determinations. Similarly two measurements of the 15° disk against the 180° disk, the shortest distance between the test Nernst and the photometer screen being about 73 cm instead of 50 cm, as was used in the previous measurements, gave as the errors $+0.22$ per cent and $+0.49$ per cent, with a mean of $+0.36$ per cent. This is only 0.09 per cent lower than the mean value of the previous determinations, but it is in the right direction.

If, now, to the above considerations we add that of the deviations of the radiation of a cylinder from the inverse square law, which are also in the right direction, we reduce the error for the small openings still further. For a cylinder 20 mm long and 1 mm radius the error for the distances used with the 10° disk would be $+0.11$ per cent. For an 88 watt Nernst 15 mm long and 0.6 mm radius the error probably would be but slightly less than $+0.11$ per cent, or about $+0.08$ per cent or $+0.09$ per cent. Similar errors would be

present in the measurements of the other disks, so that the complete curve of Fig. 11 would be lowered by a small amount, the lowering being greatest at 10° where it would amount to about 0.08 per cent, and least at 288° , where it would be approximately zero.

This lowering of the curve due to the deviation from the inverse square law combined with that indicated by the experiments described just above would seem to justify a total lowering of the curve sufficient to bring it within the limits $+0.3$ per cent and -0.3 per cent as shown in the solid curve of Fig. 11.

No attempt has been made to investigate the effect of diffraction at the edges of the disk. This might possibly affect the result to some small extent.

In order to determine whether the personal equation entered to any extent into the results obtained, several sets of readings were made by a second observer. With the 210° disk he made but one set of measurements obtaining the result $+0.04$ per cent. With the 144° opening he obtained $+0.21$ per cent; with the 120° opening, $+0.16$ per cent and $+0.20$ per cent; with the 90° opening, $+0.12$ per cent, $+0.26$ per cent, and -0.12 per cent; with the 60° opening $+0.14$ per cent; and with the 15° opening $+0.11$ per cent, $+0.29$ per cent, $+0.33$ per cent, and $+0.37$ per cent. His readings are thus in the same direction as my own and for the most part agree with them, although on the whole they show somewhat smaller deviations, particularly for the smaller openings.

(d) *Colored Light*.—In the investigation of the law with colored light it was thought sufficient to determine whether for several points on the curve the values obtained with colored light are in approximate agreement with the corresponding values for white light. The same method of observation as that for white light was used, but the error of observation was very much larger, partly because of the color and partly because of the reduced intensity which made observations on the 15° disk very difficult. The 240° , 60° , and 15° openings were used, and each was tested with red, green, and blue light by inserting pieces of glass in the eyepiece of the photometer. The red glass was ordinary ruby glass and was found on examination with a spectroscope to transmit very little other than red, although the band in the red was quite broad. The green glass was supposed to be a very high grade of monochromatic

glass, but it transmitted much light of other colors, as did also the cobalt blue glass used for the blue.

In connection with the measurements with colored light it was thought desirable to make a few experiments with color differences on the two sides of the screen as this condition always existed in Ferry's experiments when he obtained the large errors. The test lamp, as in the previous experiments, was a Nernst glower at normal current. The comparison lamp, however, was a 16 cp anchored oval, 118 volt, 3.5 watt per candle incandescent lamp burning at different voltages ranging from 116 to 102 volts. The color difference was very marked when the incandescent lamp was at normal voltage, but in order to insure the detection of any error that might exist, the incandescent lamp was burned at 102 volts in the measurements with the 15° disk. At this voltage the contrast in color was very great and yet no greater deviations than those for white light on both sides of the screen were detected.

TABLE V.

Deviations from Talbot's Law Using Red, Green, and Blue Light, and Color Difference.

	240°	Means	60°	Means	15°	Means
Red light	+0.09 % — .07	+0.01 %	+0.37 % .86	+0.62 %	+1.06 % — .25 + .52 *+0.58	+0.44 %
Green light	—0.09 +0.06	—0.02	+0.51 .43	+0.47	+0.05 .42	+0.24
Blue light	+0.01 — .13	—0.06	+0.41 .35	+0.38	+0.30 .70 *+0.47	+0.50
Color difference...	+0.21 —0.14	+0.04	+0.63 .45 .30	+0.46	+0.57 .56	+0.56

In making the measurements with color difference the absorption strips were removed from the Lummer-Brodhun photometer and the setting was made for a match, instead of for equal contrast. The same openings were used as in the experiment with the red, green,

and blue light. All of the results of the experiments with colored lights, and of those in which there was a color difference on the two sides of the screen are contained in Table V. The first column contains the four different conditions under which the experiments were made. In the second, fourth, and sixth columns are given the corresponding results, and in the third, fifth, and seventh columns are given the means of the values in the second, fourth, and sixth columns, respectively. The observations marked with an asterisk were made by the second observer and are not included in the means.

From a comparison of Tables IV and V it is seen that within the range of experimental error the deviations from Talbot's law are the same for white and colored light and for difference in color on the two sides of the screen. It would be well, however, to make further experiments on colored light in order to reduce the probable error, and obtain results more nearly comparable with the results for white light.

5. CONCLUSIONS.

The results of this investigation may be summarized as follows:

(1) Talbot's law, in its application to a rotating sector disk, is verified for white light for all total angular openings between 288° and 10° , to within a possible error of 0.3 per cent, which probably expresses the limit of accuracy of the experiments.

(2) The observed deviations from the law for red, green, and blue light are of the same order of magnitude as those for white light, and hence Talbot's law is verified for red, green, and blue light, though not to such a high accuracy as for white light. Moreover, a difference in color on the two sides of the photometer screen produces no appreciable change in the observed deviation from the law.

A NEW DETERMINATION OF THE ELECTROMOTIVE FORCE OF WESTON AND CLARK STANDARD CELLS BY AN ABSOLUTE ELECTRODYNAMOMETER.

By Karl E. Guthe.

1. INTRODUCTION.

Since our practical electromagnetic units are based upon the c. g. s. system, and in some countries are defined in terms of the units of the latter,¹ it is of the greatest importance to have the relation between the two systems determined with as great accuracy as possible. By a series of classical experiments² the absolute value of the ohm = 10 c. g. s. units has been determined with a probable error of 1 in 5,000, and based upon the value thus found a practical unit, the mercury ohm, is now in general use, and can probably be reproduced with an accuracy of 1 in 50,000.³

There remain the electromotive force and the current. The evaluation of the former in electrostatic measure and consequent reduction to electromagnetic units by means of the ratio between these two systems can not give us very reliable results so long as the above ratio has not been determined with a greater accuracy than that obtained up to the present time. It has, however, the advantage of being a direct determination—i. e., one in which no other electrical units have been used—provided the ratio of the two systems is independently determined. Electromotive force may also be measured indirectly as the difference of potential produced by a known current at the terminals of a known resistance. Its determination is thus closely related to that of an electric current,

¹ Wolff; this Bulletin, 1, 39; 1904.

² Dorn, *Zs. für Instrumentenkunde*, 28, appendix, 1893.

³ Jaeger, *Sitzungsberichte der K. P. Akademie der Wissenschaften (Berlin)*, 25, p. 547; 1903. Smith, *Phil. Trans. Roy. Soc. A*, 204, p. 114; 1904.

but introduces at the same time any error present in the absolute value of the ohm. This method leads, however, to more reliable results than the one before mentioned, and has therefore been frequently used for the determination of the electromotive force of standard cells. The problem is then to construct an apparatus which will allow a calculation of the current flowing through it from the electromagnetic effects which it produces. This same current may, of course, be used also for the determination of the electrochemical equivalent of silver or of any other suitable metal. This would be an absolute determination; it would, however, require the use of the current for the whole time the silver is deposited in the coulometer, and therefore it is simpler to determine first the electromotive force of a standard cell and then calculate the electrochemical equivalent from an independent series of experiments. This enables one to determine the relation between the two with an accuracy much greater than is obtainable with the absolute instrument. The problem of determining the absolute value of an electromotive force or the electrochemical equivalent of silver is thus reduced to the absolute measurement of current. We may divide the various methods and types of instruments proposed for this purpose into two large classes: (A) *Electromagnetic methods*, in which the action between the magnetic field of the earth and the current, or that between the current and the known moment of a magnet is measured by means of standard galvanometers or by means of the electromagnetic balance. (B) *Electrodynamic methods*, characterized by an action between two magnetic fields which are both produced by the same current. The instruments used in these latter methods are current balances and electrodynometers.

2. ELECTROMAGNETIC METHODS.

After having been proposed by W. Weber¹ for the absolute measurement of current in 1840 the tangent galvanometer has been frequently employed for this purpose, especially by Bunsen, Casselmann, and Joule, and in more recent times by Fr. and W. Kohlrausch² and by Van Dijk and Kunst.³ These last four observers have obtained

¹ Weber; Pogg. Annal. 55, 27; 1842.

² Fr. and W. Kohlrausch; Pogg. Annal. 149, 170; 1886.

³ Van Dijk and Kunst, Proc. Roy. Acad. Amsterdam, 1904, Annalen der Physik 14, 569; 1904.

very satisfactory results for the electrochemical equivalent of silver; but besides the large number of correction factors to be taken into account, the method necessitates an accurate knowledge of the horizontal intensity of the earth's magnetic field expressed in c. g. s. units and its variation during the progress of the experiment. Even with no outside disturbances H can not be measured more accurately than to about 1 in 4,000.¹ The same criticism applies to all other forms of instruments belonging to this general group—for example, the Gaugain-Helmholtz and the Gray types of the tangent galvanometer, and the sine, cosine, and bifilar galvanometers. The results with the electromagnetic balance are to be trusted still less.²

3. ELECTRODYNAMIC METHODS.

In the current balance the electrodynamic action between two coils carrying the current to be measured is balanced by known weights. The first balance constructed for this purpose was that of Cazin (1863), who determined by means of it the electrochemical equivalent of water. In England,³ France,⁴ and Germany,⁵ instruments of this type have been used, and while the French results differ considerably from those obtained in the other countries, this method is doubtless of great value. The acceleration due to gravity can be determined with an accuracy of 1 in 20,000;⁶ and with an apparatus whose dimensions are easily determined—for instance, with coils of a single layer very reliable results can be expected. Such determinations are being carried out at present at the National Physical Laboratory in England.

The electrodymanometer was proposed for absolute measurements by W. Weber, and an instrument with two stationary coils at a distance equal to their mean radius (as proposed by von Helmholtz) was constructed by the Committee on Electrical Standards of the British Association. This apparatus was first used by L. Clark⁷

¹ Bauer; *Science*, **22**, 16; 1905.

² Koepsel; *Wied. Annalen*, **31**, 250; 1887.

³ Rayleigh and Sidgwick; *Phil. Trans. Roy. Soc.* **175**, 411; 1884.

⁴ Mascart; *Journal de Physique*, **1**, 109; 1882. Potier and Pellat, *Journal de Physique*, **9**, 381; 1890. Pellat and Leduc, *Comptes Rendus*, **136**, p. 1649; 1903.

⁵ Kahle, *Wied. Annalen*, **59**, 532; 1896.

⁶ *Jahresber. Dir. Preuss. Geod. Inst.*, 1904. p. 26.

⁷ Clark, *Phil. Trans. Roy. Soc.* **164**, 1; 1874.

for the measurement of the emf. of the Clark standard cell, and a similar instrument was employed by R. O. King.¹ A type of electro-dynamometer with coils of a single layer of wire was proposed by A. Gray.² Such an instrument was constructed by Patterson and Guthe³ for the determination of the electrochemical equivalent of silver, and also used by Carhart and Guthe⁴ for the measurement of the emf. of the Clark cell. In these cases the electromagnetic effect of the two coils upon one another was balanced by the torsional moment of a single wire, whose mechanical properties were determined by preliminary experiments. In the following table are given the most reliable results obtained by absolute methods, the values for the silver equivalent being reduced to those which would have been found in a silver coulometer of the porous cup type.⁵ This correction is necessary, since different experimenters have used different types and consequently obtained results which can not be directly compared with each other. The mercurous sulphate of the Clark cells cited was prepared in the well-known chemical way, the emf. being reduced to 15° C.⁶

TABLE I.

Electrochemical Equivalent of Silver.

Observer.	Year.	Instrument.	El.-chem. Equiv.
Fr. and W. Kohlrausch	1884	Tangent galvanometer	0.0011177 gram.
Rayleigh and Sidgwick	1884	Current balance	0.0011176 "
Potier and Pellat	1890	Current balance	0.0011189 "
Patterson and Guthe	1898	Electrodynamometer	0.0011177 "
Pellat and Leduc	1903	Current balance	0.0011190 "
Van Dijk and Kunst	1904	Tangent galvanometer	0.0011178 "

Electromotive Force of the Clark Standard Cell.

Rayleigh and Sidgwick	1884	Current balance	1.4345 volts.
Kahle	1896	Current balance	1.4322 "
Carhart and Guthe	1899	Electrodynamometer	1.4333 "

The values obtained by the author in his work with Professors Carhart and Patterson showed that the electro-dynamometer method

¹ Callendar, Trans. Roy. Soc. 81, 199, 1902.

² Gray, Absolute Measurements, 2, pt. 1, 274.

³ Patterson and Guthe, Physical Review, 7, 257; 1898.

⁴ Carhart and Guthe, Physical Review, 9, 288; 1899.

⁵ Guthe, this Bulletin, 1, p. 363; 1905.

⁶ Centigrade scale is used throughout this paper.

is well adapted to work of this kind. In the former investigations, however, the actual number of determinations was small, and it seemed therefore advisable to repeat them with an instrument of improved construction and after a more thorough investigation of the various factors entering into the calculation, especially of the elastic properties of the suspension used, and the influence of irregularity of winding upon the field inside the coil.

4. THE ELECTRODYNAMOMETER.

As was first pointed out by Gray¹ the expression for the torque between the two coils of an electro-dynamometer assumes a simple form if the dimensions of both coils are chosen so that the length and the radius are in the proportion $\sqrt{3}:1$, if their centers coincide and, finally, if the dimensions of the fixed coil are large in comparison with those of the movable. Under these conditions the expression for the torque between the two coils with their axes at right angles to each other becomes

$$T = \frac{4\pi^2 N n r^3}{\sqrt{D^2 + L^2}} I^2$$

where N and n are the number of turns in the stationary and movable coils, D and L the diameter and length of the stationary, r the radius of the movable coil and I the current, expressed in c. g. s. units.

In the construction of the apparatus the conditions laid down above were closely followed. A calculation of the correction terms due to slight deviation from these conditions showed that they were entirely negligible.

5. THE STATIONARY COIL.

The frame of the stationary coil was made of plaster of Paris.² Experiments carried on in Ann Arbor, in conjunction with Professors Carhart and Patterson, and which led to the construction of a new instrument while the author was still at that place, showed the great superiority of this material over wood as used in our first instrument. If sufficient care is exercised in the preparation of the mixture of plaster of Paris and water it is easy to make cylinders

¹Gray, l. c.

²The instrument was constructed by Mr. Rudolph Hellbach of this Bureau, to whom I am indebted for valuable assistance in the solution of mechanical difficulties.

of any desired size. Plaster of Paris is, as will be shown, nonmagnetic, and while not as hard as marble (an advantage, so far as working it is concerned) it is sufficiently so for the purpose. The cylinder made at the Bureau of Standards was about 55 cm long and had a diameter of a little more than 55 cm and a wall thickness of about 10 cm. The larger air holes, from which it was remarkably free, were filled in and the whole cylinder soaked in melted paraffin before being turned. It was first turned on a large lathe, but after the cylinder was removed from the chucks it was found to have sprung and the ends were slightly elliptical. It was then finished on one of the boring mills at the United States Navy-Yard at Washington. On this machine it stands on end and is therefore not subjected to internal strains.

The diameter was measured by means of a large caliper, specially constructed of nickel iron in order to minimize the temperature changes due to handling it. The readings of the caliper were compared with those on a Brown and Sharpe steel end standard whose length was determined by the division of Weights and Measures to be 50.0055 cm at 25° with a temperature coefficient equal to 0.000011.

While still on the table of the boring mill seven equidistant pencil lines were drawn on the circumference, the outer ones being 5 cm from the edge. These circles were numbered 0, I, II, III, etc., and were divided by 24 straight lines, drawn parallel to the axis, into two sets of equal arcs of 30°. These lines were marked 0, 1, 2, 3, etc., and 0', 1', 2', 3', etc., respectively. We were thus enabled to determine the exact location of the opposite ends of a given diameter by the intersection with one of the circles of two lines 180° apart; thus 84 diameters were located and their lengths measured by three observers, who divided the work so that one of them took one set of 42 diameters, one the other set, and the third one half of each. Each observation was the mean of three readings, making the total number of readings equal to 378.

The results are very satisfactory; the mean diameters as determined by the three observers and reduced to 25°, being:

Observer	Mean Diameter
Jansky	49.9141 cm
Pierce	49.9136
Guthe	49.9135

In each circle the readings of the different diameters agreed within 0.03 mm, and the average diameter for the different circles along the cylinder shows only the very slight increase of 0.05 mm from one end to the other.

Circle	0	I	II	III	IV	V	VI
Diameter cm.....	49.9111	49.9121	49.9140	49.9137	49.9146	49.9140	49.9163

The average diameter of the plaster-of-Paris frame for the stationary coil is therefore 49.9137 cm at 25°.

In order to determine the temperature coefficient of the cylinder it was transferred to the refrigerating room of the Bureau of Standards and the mean diameter measured at a lower temperature. The thermometer was placed, as in the above experiments, in a hole drilled in the side halfway between the outer and the inner surface. On account of the temperature variations of the room the temperature indicated by the thermometer might not have been the average temperature of the cylinder; nevertheless, it gives us an approximate value for the expansion of plaster of Paris.

At 12° the average diameter was found to be 49.8976 cm, giving the coefficient of expansion of the cylinder as 0.000025. In the absolute determinations the temperatures were always in the neighborhood of 25 degrees, and hence a small error in the temperature coefficient could have no appreciable effect upon the final result.

The cylinder was then carefully wound with a single layer of double silk-covered wire of diameter equal to 0.0495 cm. Instead of a single wire, however, two separate wires were wound side by side at the same time so that each turn of one would lie between two of the other. This was done in order to enable us to determine the insulation resistance of the completed instrument by measuring the resistance between the two wires.

The measurements of the mean diameter were now repeated and resulted in a value equal to 50.0112 cm at 25°. The current flowing through the coil may be assumed to be concentrated at the center of the wire so that we obtain for the effective diameter of the stationary coil $D = 49.9624$ cm.

After the measurements had been completed the coil was given three coats of shellac.

For completeness I shall add one of the sets of measurements on 42 diameters, each being the mean of three readings, in order to show how accurately cylindrical the coil is.

TABLE II.

Showing 42 Diameters of the Cylinder.

	σ'	τ'	α'	β'	γ'	δ'
0	50.0108	50.0119	50.0098	50.0101	50.0100	50.0127
I	.0083	.0094	.0124	.0089	.0102	.0121
II	.0129	.0119	.0108	.0092	.0102	.0116
III	.0095	.0113	.0099	.0129	.0098	.0099
IV	.0089	.0116	.0109	.0092	.0113	.0090
V	.0109	.0108	.0113	.0118	.0108	.0105
VI	.0114	.0108	.0097	.0103	.0106	.0124

6. LENGTH OF STATIONARY COIL.

The total number of turns of wire on the stationary coil was 872, and although great pains were taken to make the winding as uniform as possible we did not succeed in making it perfectly uniform, since there was no lathe at our disposal big enough to hold the cylinder and besides we did not wish to subject it to new strains. The winding was therefore done entirely by hand, and the irregularity of winding was taken into account in determining the strength of the magnetic field at the center of the coil. The cylinder was placed on end, leveled, and a steel scale subdivided to 0.2 mm placed parallel to the axis so as to touch the wires along one of the lines marked on the cylinder before it was wound. Then, by means of a cathetometer, whose short-focused telescope moved parallel to the line of contact between scale and coil, the space covered by 50 turns was measured successively along the line. In this way an accurate knowledge of the irregularity of the windings along 12 lines parallel to the axis and 30 degrees apart was obtained. The results are embodied in the following table:

TABLE III.
Length of 50 Turns Along the Lines.

Number of Turns	1'	2'	3'	4'	5'	6'
0-50	24.867 mm	24.895	24.900	24.829	24.892	24.942
50-100	.909	.919	.944	.971	.924	.935
100-150	.960	.962	.965	.996	.958	25.029
150-200	.887	.814	.870	.804	.951	24.882
200-250	.921	.773	.892	.888	.864	.962
250-300	.774	.556	.780	.823	.749	.756
300-350	.600	.545	.652	.591	.641	.621
350-400	.688	.634	.730	.698	.670	.712
400-450	.680	.658	.647	.705	.733	.676
450-500	.600	.746	.614	.595	.581	.623
500-550	.527	.637	.539	.532	.511	.577
550-600	.606	.793	.517	.622	.581	.542
600-650	.724	.896	.765	.768	.788	.852
650-700	.794	.942	.791	.814	.781	.800
700-750	.911	.990	.962	.968	.949	25.029
750-800	.928	.915	.932	25.000	.940	24.965
800-850	.910	.915	.903	24.926	.889	.959
850-872	11.063		11.059	11.200	11.047	11.085

Number of Turns	7'	8'	9'	10'	11'	12'
0-50	24.985 mm	24.937	25.003	25.076	25.023	24.894
50-100	.929	.949	24.921	24.939	24.939	.909
100-150	.956	.939	.930	.950	.965	.953
150-200	.898	.821	.900	.844	.867	.841
200-250	.864	.708	.876	.815	.892	.939
250-300	.727	.703	.685	.747	.737	.758
300-350	.591	.578	.647	.603	.682	.644
350-400	.703	.697	.692	.724	.698	.732
400-450	.703	.699	.705	.764	.683	.668
450-500	.632	.683	.733	.865	.657	.598
500-550	.574	.681	.632	.624	.632	.597
550-600	.580	.728	.665	.547	.552	.549
600-650	.690	.878	.849	.735	.744	.762
650-700	.812	.922	.780	.768	.841	.800
700-750	.983	.956	.997	25.009	.974	25.017
750-800	.950	.995	.959	24.927	.937	24.977
800-850	.950	.968	.965	.959	.962	.953
850-872	11.070		11.188	11.053	11.025	11.044

As will be seen from this table the measurements on the 12 lines show a close agreement as to the distribution of the turns along the coil. This comes out very distinctly in plotting the 12 different values for the corresponding numbers and it is therefore admissible to take the mean of all to represent the irregularity of winding. This gives us Table IV and the broken curve in Fig. 1 where the space occupied by 50 turns is plotted as a function of the distance from one end.

The winding is somewhat closer in the middle than at the ends, the curve being to a certain extent symmetrical about the center.

The value found by measurement for the average length of the coil has to be corrected for the error of the steel scale between the

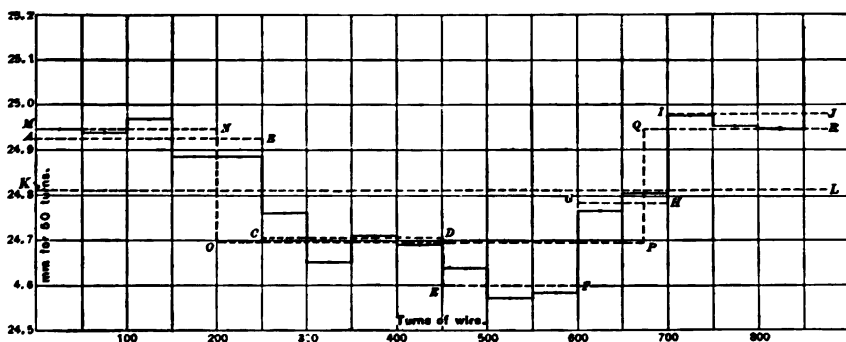


Fig. 1.—Irregularity of Winding of Stationary Coil.

two points used in these measurements. The correction was found by the division of Weights and Measures to be $+0.057$ mm at the temperature at which it was used. The average length of the coil is therefore at 25° $L = 43.2764$ cm. The length calculated from the theoretical relation between length and radius would be 43.32 cm.

In correcting for the irregularity of winding I substituted for the actual coil two layers of uniform winding, superposed one upon the other, the first to be of the same average length as the coil, the second, a shorter one, at the center. As the length of the latter the distance between two points where the density of the winding considerably increased was selected. The short coil was assumed to have a length equal to the distance between the 200th and the 672d turns, i. e., the density of winding of the longer coil to be the average of that of the 400 outer turns represented by MN and QR , Fig. 1. This average gives for the space covered by 50 turns

24.946 mm and the total number of turns in the longer coil $(432.764 \div 24.946) \times 50 = 867.40$ turns. This leaves 4.60 turns for the smaller coil, distributed over 233.196 mm; or 872 turns in all.

TABLE IV.
Mean Length of 50 Turns.

Number of Turns	Length of 50 Turns	Corrected
0-50	24.941 mm	24.945 mm
50-100	.936	.940
100-150	.967	.970
150-200	.884	.887
200-250	.884	.887
250-300	.755	.758
300-350	.649	.652
350-400	.706	.709
400-450	.693	.696
450-500	.636	.639
500-550	.572	.575
550-600	.585	.588
600-650	.763	.766
650-700	.801	.804
700-750	.975	.978
750-800	.949	.953
800-850	.942	.946
850-872	11.069	11.071
0-872	432.707	432.764

The strength of the magnetic field at the center of the coil is expressed by the formula

$$H = \frac{4\pi NI}{\sqrt{D^2 + L^2}} = CI$$

where N is the number of turns, D the diameter and L the length of the coil, and in the following the constant C for each of the two coils is given.

$$\begin{aligned} C_1 \text{ due to the long coil} &= 164.905 \\ C_2 \text{ due to the short coil} &= 1.048 \\ \hline \text{Sum} = C &= 165.953 \end{aligned}$$

That a correction of this kind is necessary becomes quite apparent if we work out the value of C under the supposition that the 872 turns of wire are uniformly distributed over the average length of 43.2764 cm. In this case we obtain $C = 165.778$ or a value in error by more than 1 in 1,000.

In the above calculations no account was taken of the fact that the short coil does not fulfill the conditions leading to the simplified formula used for the evaluation of C , but a calculation using the more complicated formula as given by Patterson¹ shows that the correction terms are so small that they will not enter in the first six significant figures of the constant C . The value of the correction represented by this second coil of 4.6 turns is somewhat uncertain, inasmuch as the winding is not exactly symmetrical about the mean plane and the correction turns should not be uniformly distributed over the length of the assumed uniform winding.

Professor Rosa² has calculated the resultant magnetic field at the center of the coil by computing the effects of the 18 sections of the winding separately and adding the results. This gives an exact value of C on the assumption that the winding of each section is uniform. There can be very little uncertainty in the value of C thus found, namely, 165.992, due to small variations in the winding of the separate sections.

Rosa has checked this value of C by assuming a uniform distribution of the 872 turns over the cylinder and superposing on this five current sheets to represent the irregularity of winding. Two of these are negative and three are positive, the magnetic effect of the positive current sheets being in excess of the negative by 0.202. This added to the effect of the uniform winding, 165.778 gives 165.980, a value agreeing with that found by considering the effect of each section separately within one part in 14,000. The more exact value of C , 165.992, is used in the calculations of this paper.

The five current sheets which represent the irregularity of winding are as follows:

(1) over the first 5 sections or 250 turns, carrying a current in the negative direction of 1.124 (assuming unit current in the wire) and hence equivalent to -1.124 turns, AB of Fig. 1, 50 turns equal to 24.926 mm;

¹ Patterson, *Physical Review*, 20, 309; 1905.

² See his paper in this Bulletin.

- (2) over four sections (250–450 turns) carrying a current of 0.890, in the positive direction, *CD* of Fig. 1, 50 turns equal to 24.704 mm;
- (3) over three sections (450–600 turns) carrying a current of 1.292, in the positive direction, *EF* of Fig. 1, 50 turns equal to 24.601 mm;
- (4) over two sections (600–700 turns) carrying a current of 0.119, in the positive direction, *GH* of Fig. 1, 50 turns equal to 24.785 mm;
- (5) over the last three sections (700–872 turns) carrying a current of 1.177, in the negative direction, *IJ* of Fig. 1, 50 turns equal to 24.985 mm.

Thus the two negative current sheets were together equivalent to -2.301 turns and the three positive current sheets were equivalent to $+2.301$ turns. The magnetic effect of the first is -0.340 , of the second is $+0.542$, the difference being $+0.202$ to be added to the value due to the assumed uniform winding, as stated above. The only other irregularity of winding, besides the nonuniformity, is the slight displacement of four wires, two on each side of the hole through which the suspension for the movable coil passes. In our dynamometer this hole was only 3 mm wide and 8 mm long in the direction of the winding. It was necessary to place the wires which would otherwise have passed over it on top of the adjacent wires for a distance of about two centimeters. A very thin hard rubber lining of the hole projecting two millimeters above it allows the winding to extend close to it. The effect of this displacement upon the magnetic field strength at the center is entirely negligible.

7. THE MOVABLE COILS.

In order to increase the accuracy of the result it was decided to use for these determinations two movable coils of different dimensions. The frames for both consisted of porcelain cylinders from the Königlische Porzellan Manufactur at Berlin, accurately ground in the instrument shop of the Bureau. Their average diameters were carefully measured by a Zeiss vertical comparator having a calibrated scale, and by means of a newly constructed end comparator. The values obtained by the two instruments were in sufficient agreement. The method employed to locate the exact position of the diameters was the same as described under the measurements of the stationary coil; 42 diameters on the larger cylinder were measured, 36 on the smaller. As before three independent sets of readings were taken

TABLE V.

Diameters of the Larger Coil: $t=23^{\circ}8$.

Number of Circle	0	1	2	3	4	5
0	99.3267 mm	99.3303	99.3277	99.3282	99.3273	99.3293
I	.3358	.3322	.3326	.3305	.3307	.3328
II	.3377	.3373	.3342	.3326	.3320	.3369
III	.3393	.3365	.3337	.3305	.3340	.3363
IV	.3372	.3368	.3322	.3328	.3340	.3390
V	.3381	.3332	.3317	.3306	.3358	.3370
VI	.3319	.3292	.3234	.3262	.3295	.3354

by as many observers whose values for the average diameter agreed within 0.002 mm. In reducing the diameter to a temperature of 25° the temperature coefficient of Berlin porcelain was taken as 0.000004.

TABLE VI.

Diameters of the Smaller Coil: $t=23^{\circ}8$.

Number of Circle	0	1	2	3	4	5
0	75.2203 mm	75.2172	75.2005	75.2169	75.2213	75.2270
I	.2217	.2130	.2129	.2145	.2220	.2235
II	.2184	.2121	.2165	.2170	.2196	.2234
III	.2175	.2118	.2128	.2143	.2197	.2197
IV	.2128	.2111	.2189	.2158	.2177	.2198
V	.2076	.2007	.2027	.2071	.2128	.2104

Tables V, VI, and VII show that both cylinders were quite accurately ground, the largest difference of any reading from the mean amounting in the larger one to 1 in 10,000, and in the smaller to 1 in 4,000. The variation is mainly due to the fact that the ends are somewhat rounded off, i. e., the diameter is smaller, where in the complete coil no wire was wound. Taking the average for the different circles it is seen that the maximum variation from the mean is in the larger coil only 0.006 mm, and in the smaller 0.008 mm. Here also all but the outer circles agree within 0.002 mm.

TABLE VII.

Average Diameters at 25° C.

Circle	Larger Coil	Smaller Coil
O	99.3288 mm	75.2176 mm
I	.3329	.2183
II	.3356	.2182
III	.3355	.2164
IV	.3359	.2164
V	.3349	.2073
VI	.3298	
Average	99.3333	75.2157

The cylinders were then carefully wound with bare copper ribbon 0.375 mm thick, a space of about 0.6 mm being left between consecutive turns. Measurement of the diameters after winding showed that the ribbon had become thinner during the winding by 0.008 mm and 0.0053 mm, respectively, due to the tension under which it was put on. The total average for the effective diameter of the cylinders, i. e., the diameter of the porcelain cylinder, plus one thickness of wire, was found to be at 25°,

For the larger cylinder . . . 99.702 mm.

For the smaller cylinder . . . 75.5854 mm.

The length of the movable coil does not enter explicitly into the calculation. The number of turns were:

Larger cylinder, $n = 109$.

Smaller cylinder, $n = 83$.

The area of the movable coil multiplied by the number of turns, i. e., the effective area of the coil is expressed by

$$A = \pi r^2 n, \text{ where } r \text{ is the radius of the coil,}$$

and, calculated from the above data

For the larger coil $A = 8509.9 \text{ cm}^2$, and

For the smaller coil $A = 3724.3 \text{ cm}^2$.

Finally three coats of shellac were applied.

8. ELECTRODYNAMOMETER COMPLETE.

In the construction of the electro-dynamometer as a whole (see Fig. 2) care was taken to avoid any additional field which might produce a torsional moment upon the movable coil. The terminal binding posts of the two wires on either end of the stationary coil were placed in the same straight line parallel to the axis. They were fastened to a hard rubber block screwed into the plaster of Paris by means of brass screws. The wires were continued to the straight line and then bent at right angles, thus leading to the binding posts. They were held permanently in this position by the hard rubber blocks. Since in all experiments the current was sent through the two stationary wires in series, the two inner binding posts were connected by a straight wire. The current on leaving the second stationary coil was carried along another straight wire parallel to the axis back to the leading-in wire, where the two were twisted together until they reached at a distance of about 60 centimeters a small commutator. One of the leads was connected to one of the mercury cups of the commutator, while the other continued to the battery. The two wires leading to the movable coil started from two of the mercury cups and were twisted until they reached the mercury cups of the movable coil to be described presently. The fourth mercury cup connected with the battery circuit. By means of this commutator the current in the movable coil could be reversed while the direction in the stationary coil remained the same. By means of another commutator, several meters distant, the current through the whole instrument could be reversed. This gives four possible combinations for the direction of the current in the instrument. In the battery circuit was inserted the standard resistance from the terminals of which two potential wires led to the potentiometer. The latter apparatus could be connected either to the resistance or to the standard cells whose emf. was to be measured. In order to be able to bring the movable coil to rest rapidly the instrument was shunted by a circuit containing a dry cell and a high resistance. By judicious working of the key in this circuit the amplitude of the vibration could in a very short time be reduced to a couple of millimeters on the scale. The connection with the movable coil was made by means of two mercury cups arranged one below the other. The current was carried to the upper

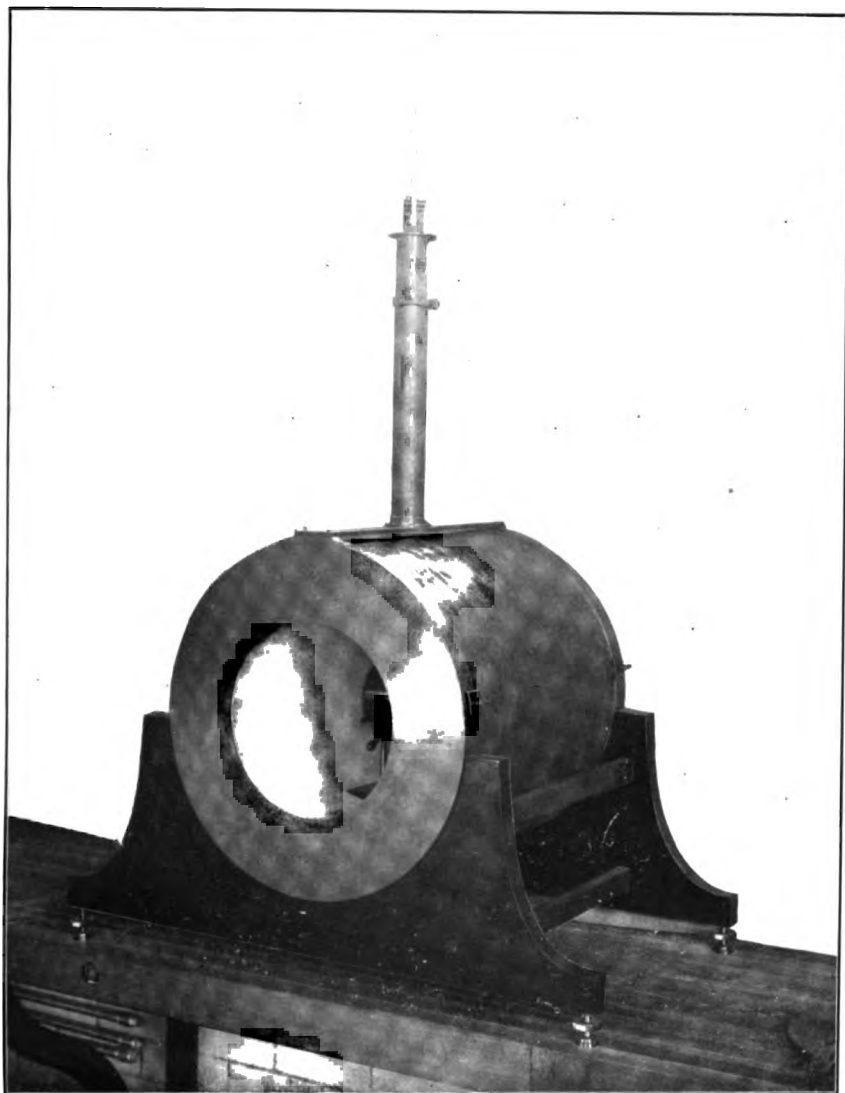


Fig. 2.—*Electrodynamometer Complete.*



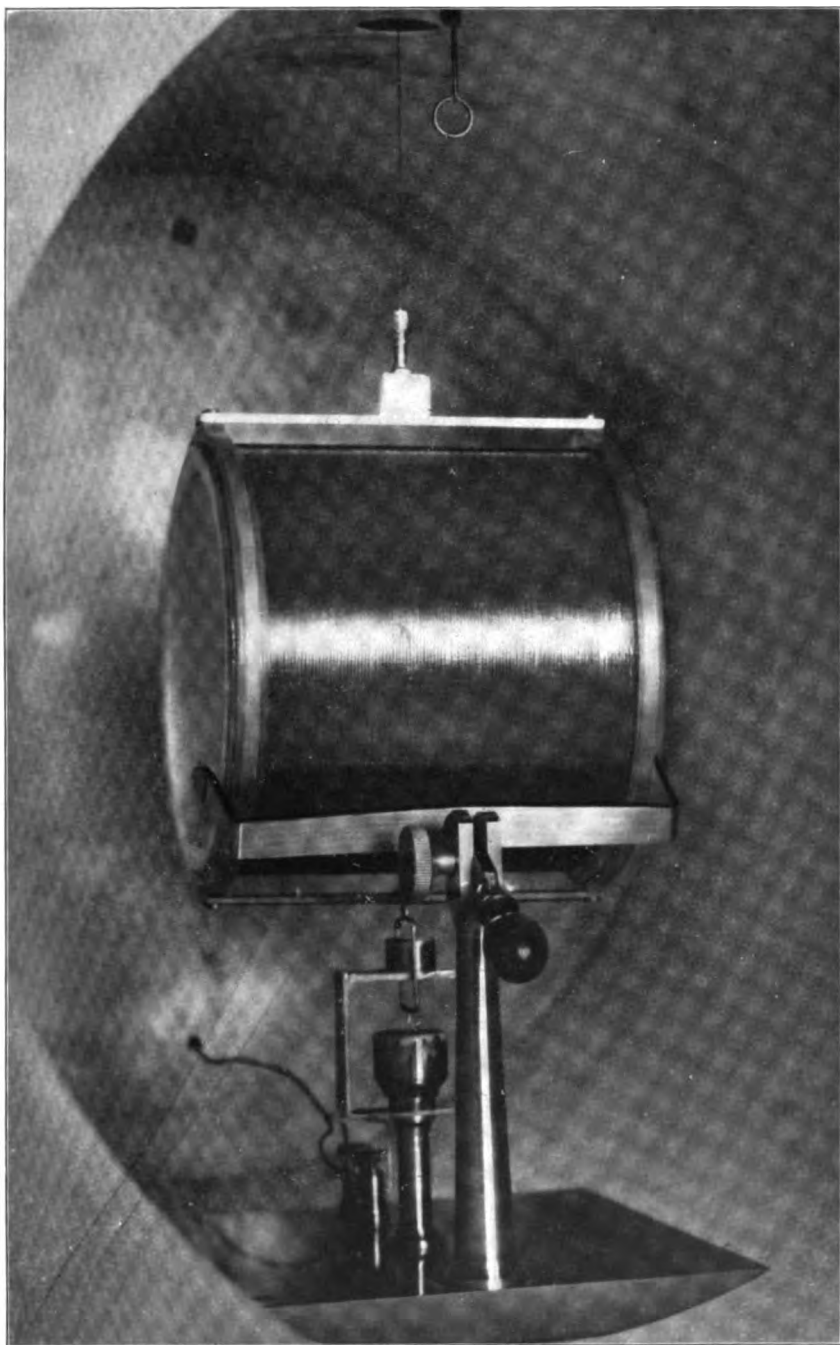


Fig. 3.—*Mercury Cup Connections for Movable Coils.*



cup through a tube surrounding the conductor leading to the lower cup. A little below the latter the tube divided, forming a fork (see Fig. 3); the upper cup was carried by a bridge which closely fitted the two prongs of the fork, but could be removed to allow the introduction of the similar loop of the movable coil.

The ends of the wire of the movable coil were held in position by narrow rings of fiber which closely fitted the porcelain cylinder and carried on the lower side a thin strip of wood along which the wires were led back to the center, one being bent at right angles, the vertical part ending in a fine platinum wire dipping into the upper mercury cup, the other forming a fork extending into a vertical loop, which latter carried the platinum wire for the lower cup. In this way dissymmetry of the circuit was avoided as much as possible. The fiber rings also served to hold the small aluminium plate on top for the connection with the suspending wire. In order to limit the motion of the movable coil to a small arc and to prevent its turning beyond its position of stable equilibrium¹ a fork was used having a width between the prongs only 3 mm larger than the length of the coil. This greatly increased the rapidity with which a determination could be made.

The suspension was supported from the top plate, capable of rotation, of a long brass tube standing on a plate of the same metal, the latter resting on the large caps which closed the plaster of Paris cylinder on either side. The whole instrument was supported by a mahogany cradle on leveling screws. The temperature was read by means of three thermometers, one inserted in the top of the brass tube, one at the bottom and read through a window in the tube, and one in the interior of the stationary coil.

It is of importance that the centers of the two coils should be coincident. For this purpose small aluminium plates with circular projections of the same diameter as the coils were slipped over the open ends of the movable coil. These plates were perforated at their centers; the straight line defined by these holes is the axis. The cap entirely closing the back of the stationary cylinder contained a vertical slit with a crosswire at the center and the circular door in front had a small hole at the center. By rotating, raising,

¹ Patterson and Guthe, l. c., p. 268.

or lowering the movable coil, and finally tilting it by means of a weight placed inside of the porcelain cylinder, the latter could without much trouble be centered very accurately. The cross plates on the movable cylinder were then removed and the torsion head rotated by 90 degrees. After this the apparatus was ready for a determination.

9. MEASUREMENT OF ANGLE OF ROTATION.

To insure a rotation of very nearly 90 degrees and the accurate determination of this angle, the following scheme was employed: On the movable and accurately fitted top of the brass tube of the electro-dynamometer a small table was placed which could be leveled

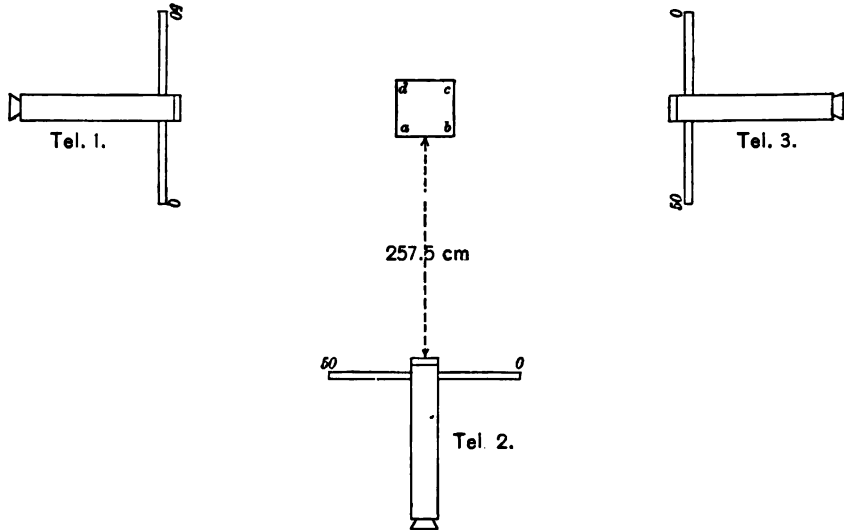


Fig. 4.—Arrangement for Measuring Angles of Glass Cube.

by three adjusting screws and which carried a glass cube with silvered sides. Opposite three sides and at a distance of 257.5 cm from the center of rotation, three telescopes and scales were set up at the same level as the mirrors (see Fig. 4). As reference point, the central scale mark 22.5 of the middle telescope, telescope 2, was chosen. The cubical mirror was carefully leveled so that on rotation the image of the scale always appeared at the same level in the field of the telescope, and then the readings in the telescopes were taken with each of the four faces toward telescope 2. The angles of the cube were marked *a*, *b*, *c*, *d*.

TABLE VIII.

Calibration of Glass Cube.

Position of Cube	Reading in—		
	Tel. 1	Tel. 2	Tel. 3
d c	24.745	22.5	24.75
a b			
a d			
b c	24.38	22.5	24.715
b a	24.445	22.5	25.16
c d			
c b			
d a	24.02	22.5	24.415

From this follows that the reading of telescope 1 from a mirror at right angles to the one pointing toward 2 would have been 24.40 and the reading of telescope 3, 24.76, a smaller reading in the latter denoting a larger angle. The corrections for the angles in terms of scale parts are given in the following table:

TABLE IX.

Correction in Scale Parts.

Angle	Tel. 1	Tel. 3	Average
a	0.345	0.345	0.345
b	-0.02	0.01	-0.005
c	0.045	0.045	0.045
d	-0.38	-0.40	-0.39

Calculating the values of the angles in degrees we obtain

Angle *a* equals 90.0382 degrees.

b 89.9994

c 90.0050

d 89.9566

Angles *b* and *c* being so close to 90 degrees, they were used in all the following experiments.

10. MAGNETIC TESTS.

Before deciding on the materials to be used in the construction of the electro-dynamometer it was necessary to test their magnetic qualities. For this purpose two induction coils were constructed which allowed the insertion of a block of the material to be tested inside of the primary. The primaries consisted of 100 turns of No. 14 B. & S. wire and the secondaries of 2,000 turns of fine wire. The primaries were connected in series and the secondaries in series with a ballistic galvanometer so that the electromotive forces induced in them due to the making or breaking of the primary current were opposed to each other. The frames of the secondaries could be slightly displaced relatively to the primary by means of adjusting screws and the differential effect, as shown by the galvanometer, reduced to zero. After two more turns had been added to one of the secondaries a reversal of 10 amperes in the primaries produced a deflection of 40 mm of the galvanometer. A deflection of 1 millimeter corresponds therefore to a change in the number of lines of force equal to 1 in 40,000. A block of plaster of Paris, having a square cross section of 100 cm² and a length of 15 cm and snugly fitting inside of the primaries was made at the same time and of the same material as the frame for the stationary coil. A similar block of marble made of the same material as two cylinders which were intended for future work was also tested. Neither material when introduced in the primary showed an effect of more than 0.25 mm, if any, so the number of lines of force was not changed by more than 1 in 100,000 by substituting it for air. The porcelain cylinders gave the same negative result.

Fiber, brass, and aluminum, used in small quantities in the construction of the instrument, show strong magnetic effects when hung on a silk fiber near the pole of an electromagnet, but their permeability does not differ from 1 by more than 0.0001. In the small quantities in which these materials were made use of they could not possibly influence the result appreciably.

There was, however, one magnetic effect that had to be taken into account, i. e., that of the iron brackets supporting the wooden wall table on which the electro-dynamometer was placed. The magnitude of this effect was determined as follows: A current producing a deflection of the movable coil of 90 degrees was kept constant

during the test. The readings on the movable coil were taken repeatedly with and without the addition of another bracket of the same size and material in the same relative position to the dynamometer as each of the stationary brackets. It was found that only the nearest one of the latter affected the current. When the movable bracket was placed in position the deflection increased by 3 millimeters, which corresponds to an increase of the effect of the square of the current of 1 in 2,700 or of the first power of the current equal to 1 in 5,400. This effect will therefore be taken into account in the calculation of the final result.

11. SUSPENSION.

A great deal of time was spent in the attempt to find a suspension for the movable coil which would show the smallest elastic after-effect and at the same time not change its elastic properties in course of time. The attempt has been only partially successful.

The elastic after-effect was tested in a specially constructed apparatus, practically a small electro-dynamometer, whose movable coil could be brought to rest in a short time, usually a fraction of a minute. After the original zero point had been determined, a twist of 180 or 360 degrees was given to the torsion head, and after a number of minutes, varying in the different experiments between one and five, the torsion head was turned back and the new resting point of the coil and its change in course of time observed. The torsion head carried also a mirror which was observed by telescope and scale so that correction might be made for failure to bring it back to exactly the same position as at the start. It is unnecessary to relate here the large number of experiments made with different materials, some of which were treated in different ways. It suffices to say that I experimented first with substances which are known to have small elastic after-effect, namely, fused quartz, steel, platinum-iridium, and carbon. While it is true that the after-effect in all of them is small (though by no means negligible) when used in sizes sufficiently large to be able to carry the movable coil they were found to be entirely unsuited for the present purpose; when tested for their coefficient of torsional elasticity they showed considerable variation of the time of vibration under a constant stress on successive days. The only materials which at first were thought to be

suitable were straight steel wires which Dr. Weston kindly sent me. These wires, after hardening and annealing at a dull blue heat, seemed very satisfactory, but although the time of vibration seemed fairly constant with a given load while it remained on the wire, it changed considerably, when the load was removed and later attached again. Professor Patterson and I had been using a phosphor bronze suspension which seemed to have the desirable qualities, though we had not made exhaustive tests in this connection; our experience at that time did not show any such irregularities as were the regular occurrence in the early part of this investigation. My first experiments on phosphor bronze ribbons of various sizes were very unsatisfactory, on account of their large elastic after-effect. Fortunately through the kindness of Mr. H. B. Brooks I obtained a piece of phosphor bronze wire twelve years old, and this proved to be superior to all other material tried. The after-effect was small, and while it changed its elastic properties in course of time it remained constant enough for a week, so that with the proper precautions the experiments could be performed before a change took place. After the torsion head had been kept twisted 180 degrees for 2 minutes, the displacement of the zero point amounted to only 3 or 4 mm on a scale $1\frac{1}{2}$ meters from the mirror.

The modulus of torsion of the suspension, i. e., the torsional moment of the wire for unit angle, was determined by vibrating a cylinder of known moment of inertia and calculating from the formula

$$\tau = \frac{4\pi^2 K}{T^2}$$

where K is the moment of inertia of the vibrating mass and T the period of a complete vibration.

The ends of the wire were soldered into brass cylinders, the smaller of which was a pin, by means of which the wire was clamped to the torsion head. The larger cylinder, consisting of Tobin bronze, which seems to be of more uniform density than ordinary brass, formed a part of the vibrating system. It may be considered as two coaxial cylinders, the upper longer one of uniform diameter and the lower a flat circular plate of slightly larger diameter. To the lower end a small mirror was fastened. In the electrodyna-

momometer the wire was inverted, the larger cylinder passing through the torsion head while the movable coil was fastened to the small pin. Thus the length of the wire remained the same as in the torsional experiments.

Over the Tobin bronze cylinder larger hollow cylinders of different dimensions and of the same material could be slipped, the holes in these being fitted as exactly as possible to the narrower part of the former. These cylinders were very accurately turned. Their dimensions were measured by means of an end comparator and their

TABLE X.

Tobin Bronze Cylinders.

Hollow Cylinders. Temp. 25°			
Cylinder	Mean Diameter	Mean Height	Mass (incl. gold) M_1
A	44.2816 mm	38.1925 mm	483.9432 grams
C	59.6630	20.8442	483.9968
D	57.2224	15.8071	337.2682
E	49.8946	20.8864	337.6404

Inner Cylinder. Temp. 25°				
Mean Total Length $L + l$	Mean Diameter, d	Mean Height of Head, l	Mean Diameter of Head, D	Mass M_2
34.470 mm	6.049 mm	3.993 mm	7.995 mm	9.0500 grams

Glass Mirror.		
Length, a	Thickness, b	Mass M_3
14.043 mm	0.3376 mm	1.865 grams

masses accurately determined. For these determinations also I am indebted to the division of Weights and Measures. In order to avoid changes in weight due to oxidation the cylinders were gold plated, the amount of gold deposited being only a few milligrams and having, as calculation showed, a negligible effect upon the moment of inertia determined under the supposition that the cylinders consisted of bronze of uniform density. Two pairs of cylinders

were used in the final experiments, those of each pair having the same mass, but different dimensions. One of them, A, fulfilled the condition that the ratio of length to radius should be as $\sqrt{3}:1$, in which case a slight error in the axis of rotation produces a minimum error in the moment of inertia.¹ The other cylinders were flatter and had a correspondingly larger diameter. Two cylinders were made for each set in order to test whether the moment of inertia of such small cylinders could be calculated with sufficient accuracy. Two sets of different mass were made because the masses of the movable coils differed considerably and it was found that the modulus of torsion depended upon the tension of the wire. It was, in fact, necessary to make the weights of the movable coils equal to those of the cylinders with which the torsional moments were determined. This was done by adjusting the mass of the small metal rod, which was placed inside the porcelain cylinder and served for the centering of the axis, until the total weight of the movable coil was equal to that of the corresponding bronze cylinders.

Other Tobin bronze cylinders of different mass were used, but since they do not directly concern the determination under discussion, the results obtained with them will be omitted.

The moment of inertia of the hollow cylinders was calculated from the formula

$$K_1 = M_1 \frac{R^2 + r^2}{2}$$

that of the inner cylinder from

$$K_2 = \frac{M_2}{8} \left(\frac{Ld^4 + lD^4}{Ld^2 + lD^2} \right)$$

where L , d , and l , D , are the lengths and diameters of the narrower and the wider part, respectively; and that of the mirror from

$$K_3 = M_3 \frac{a^2 + b^2}{12}.$$

Table XII shows the change of the elastic coefficient in course of time. During the first experiments in July and the early part of August, it was noticed that the change was especially pronounced after the heavier cylinders had been on the wire, so after the second set in August a load heavier than any to be used in the later

¹ Limb, *Comptes Rendus*, 114, 1057; 1892.

experiments was hung on the wire. This produced a large change, but had the effect of making the wire more constant. The table also shows the agreement between the torsional moments, as determined by means of the two cylinders A and C. Of the last two determi-

TABLE XI.

Moments of Inertia of Cylinders at 25°.

	A	C	D	E
Moment of outer cylinder, K_1	1208.321	2175.730	1395.870	1066.137
Moment of small inner cylinder, K_2 ..	0.471	0.471	0.471	0.471
Moment of mirror, K_3	0.324	0.324	0.324	0.324
Total moment, K	1209.12	2176.52	1396.66	1066.93

nations, made in September, the first was taken before and the second after the determination of the electromotive force of the standard cells, showing a sufficient constancy during that interval.

TABLE XII.

Modulus of Torsion of Suspension with Cylinders A and C. Temperature 25°.

Date	Cylinder	Period	Modulus	Average
July 25-26	A	12.4136	309.766	309.755
27-29	C	16.6556	309.744	
Aug. 17-18	A	12.4176	309.567	309.569
19-21	C	16.6602	309.572	
Sept. 23	A	12.4281	309.044	309.030
26	C	16.6752	309.016	

The period was determined by the well-known method of coincidences; the second clicks being given by a break-circuit chronometer, which was regularly compared with the standard Riefler clock. The corrections of the chronometer were negligible. The temperature coefficient is rather large for the phosphor bronze, so the readings of two or three thermometers placed along the wire were carefully noted every fifteen minutes. The apparatus in which the swinging system was suspended was protected from air currents, but it was

found unnecessary to swing the cylinders in vacuo, since the friction of the air did not produce an appreciable effect on the period. Experiment showed that the periods in air and in vacuo were the same within a few ten-thousandths of a second. The periods were always determined by at least two observers and their results agreed usually within 0.0002 second.

The agreement between the torsional moduli, as calculated from the periods with cylinders D and E, was not nearly as satisfactory, as will be seen from Table XIII.

TABLE XIII.

Modulus of Torsion of Suspension with Cylinders D and E. Temperature 25°.

Date	Cylinder	Period	Modulus	Average
July 29-31	D	13.3488	309.435	309.380
Aug. 1-2	E	11.6692	309.325	
21-23	E	11.6742	309.059	309.112
24-27	D	13.3546	309.166	
Sept. 18	E	11.6809	308.634	308.688

Cylinder D gives a torsional modulus higher by 0.108 than E. At first it seemed best to throw out the value obtained with D, on account of the disk-like shape of the cylinder and the consequent liability to have too small a moment of inertia if the axis is slightly tilted, and therefore only E was used at the time of the electrical determination. This seems, however, to be an unjust discrimination against D, especially since no such tilting was noticeable to the eye. I decided therefore to use in the later calculations the mean value, i. e., one 0.054 higher than the value given by cylinder E. As can be seen from a later formula, an error of that amount would produce in the final result for the current an error a little smaller than 1 in 10,000.

The temperature coefficient for the vibration periods was determined a great many times, and was found to be 0.000195. This value agrees very well with that found by Professor Patterson and myself. The periods for a given cylinder plotted as a function of the temperature give a straight line. In the evaluation of the tem-

perature coefficient of the modulus of torsion we have, however, to take into account the expansion of the cylinders, whose coefficient was assumed to be 0.000018. Since the modulus of torsion is pro-

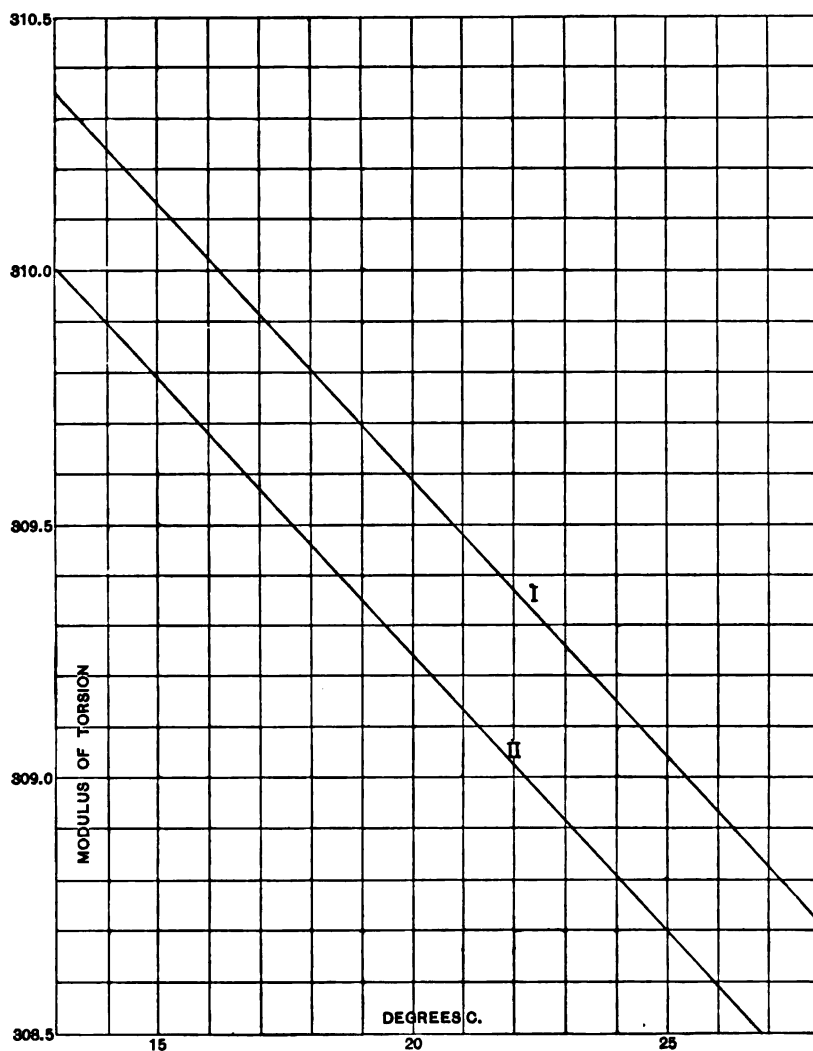


Fig. 5.—Modulus of Torsion of Suspension as a Function of Temperature. I, with larger coil; II, with smaller coil.

portional to the moment of inertia and inversely proportional to the square of the time of vibration, the *temperature coefficient for the*

modulus of torsion becomes 0.000354. In Fig. 5 the two moduli have been plotted as functions of the temperature, the higher curve corresponding to a load equal to the larger movable coil and the lower to that of the smaller coil.

12. THE STANDARD CELLS.

Professors Carhart and Hulett kindly furnished me twelve standard cells, three Clark and nine Weston (cadmium), prepared in different ways. They will be designated by the marks they bore when I received them. They had been transported carefully so as to disturb them as little as possible. All of them were of the H form and hermetically sealed by drawing out the upper portions of the tubes through which the materials had been introduced.

1. *Cadmium Cells.*— C_3 . Hg_2SO_4 prepared chemically by adding dilute H_2SO_4 to $HgNO_3$. The precipitate was washed with water until the top was quite yellow; this layer was removed and the rest made up into the paste. Kahlbaum's c. p. $CdSO_4$ used. Amalgam 12.5 per cent electrolytic cadmium melted with distilled mercury. Set up November 21, 1903, by Hulett; emf. apparently constant since January, 1904.

E_1 and E_2 . Hg_2SO_4 prepared electrolytically, and cells set up January, 1904, by Carhart. In E_1 the Hg_2SO_4 had worked in between the mercury and the platinum wire, and no results could be obtained with it.

F_1 and F_2 . Hg_2SO_4 prepared electrolytically and washed with $CdSO_4$ solution. The acid used in the preparation was of greater than molecular concentration, a condition necessary to avoid hydrolysis.¹ The $CdSO_4$ was recrystallized and the amalgam made electrolytically. Set up by Carhart and Hulett February 15, 1904.

K_6 . The electrolytic Hg_2SO_4 was washed with alcohol and ether. Set up by Hulett July 9, 1904.

K_{10} . The Hg_2SO_4 was prepared chemically by adding $HgNO_3$ to H_2SO_4 of concentration 1 to 6. Set up July 9, 1904.

O_1 and O_2 . Hulett's electrolytic Hg_2SO_4 (No. A) washed with $CdSO_4$ solution. Set up by Mr. Trout February 11, 1905.

2. *Clark Cells.*— R_1 , R_2 , and R_4 . The mercurous sulphate was prepared electrolytically and washed with zinc sulphate solution. The

¹ Hulett, Zs. für phys. Chemie, 49, 483; 1904.

zinc sulphate was twice recrystallized from Merck's c. p. salt; 10 per cent amalgam used for negative electrode. Set up by Hulett May 3, 1905. It should be stated that the electrolytic mercurous sulphate in the different series was from different batches, prepared at different times: for the E series by Carhart, for the rest by Hulett. The cells were placed in a thermostat of the Ostwald type, the cells being immersed in an oil bath and their temperature kept constant within 0.01 degree. I am much indebted to Professor Hulett for valuable help in the construction of the thermostat and the first few comparisons of the cells, which showed that the cells that he kindly left with me had not changed during transport with respect to the total average of all his cells, which were also measured at the same time.

Besides these standard cells I had at my disposal six Weston cells (unsaturated), which Dr. W. A. Noyes had a short time before brought from Europe and for which we received Reichsanstalt certificates. These cells, with the exception of one, showed no variation in their relative emf., and might therefore be used with a fair degree of accuracy for a direct comparison with the Reichsanstalt values. In fact, in making my comparisons I selected one of them (No. 813) as reference standard as having an emf. of 1.01882 volts at 21°, and a temperature coefficient of -0.00001 . It should be

TABLE XIV.

Comparison of Standard Cells. Weston Cell No. 813 assumed to have an emf. at 21° of 1.01882 Volts.

Cell	Sept. 11	Sept. 17	Sept. 24	Sept. 26
F ₈	1.01824 v.	1.01825	1.01825	1.01825
F ₉	26	26	26	25
E ₂	28	29	29	30
K ₆	31	31	23	28
K ₁₀	31	31	30	32
O ₁	31	32	31	32
O ₂	30	31	31	31
C ₃	54	55	54	55
R ₁	1.42034	1.42038	1.42036	1.42037
R ₂	35	38	36	37
R ₄	35	38	36	38

kept in mind that the Reichsanstalt always refers to the emf. of the Clark cell as 1.4328 volts. Table XIV gives the results of the comparisons, beginning with September 11, 1905. The temperature of the earlier experiments was 0.10° too high. After the thermometer (Golaz No. 3583) reading to 0.02° had been compared with the primary standards, the temperature of the thermostat was adjusted to 25° within 0.01° and kept there for the rest of the time. The comparisons were made with a calibrated Wolff potentiometer.

As will be seen, the cadmium cells with electrolytic mercurous sulphate form two distinct series, the *F* series having an emf. about 0.00006 volt lower than the rest, probably due to the fact that in the former the acid in which the mercurous sulphate was formed had a concentration larger than normal. It is rather interesting to note that cell K_{10} , prepared chemically by adding the mercurous nitrate to the sulphuric acid, also shows the low values of the electrolytic cells. The abnormal drop of the emf. of K_6 is probably due to an accidental short circuit. Cell C_1 , prepared in the usual way, has an emf. 0.00030 volt higher than the *F* series. Reducing to 20° , using the temperature coefficient found at the Reichsanstalt,¹ its value would be 1.01875 volts, or 0.00015 volt higher than the value given by them to the cadmium cell.

The emf. of the Clark cells reduced to 15° , using the accepted temperature coefficient, would be 1.43300 volts; but it seems from Hulett's² experiments with these cells that they have a slightly different coefficient, and assuming this the emf. would be 1.43293 volts at 15° , or 0.00013 volt higher than the Reichsanstalt value. Since practically the same difference had been found for the cadmium cells, prepared chemically, the conclusion seems justified that in the case of the Clark cells the electrolytic preparation of the mercurous sulphate does not produce as large a difference in the emf. as in the cadmium cells, a point which ought to be studied more carefully. The difference between the Reichsanstalt values and mine may be accounted for by a change of the Weston cells, which were newly set up, and as the account from the European Weston Electrical Company had shown changes just previous to their shipment. The low values of the Clark cells in the first com-

¹ Jaeger and Kahle, *Zs. für Instrumentenkunde*, **28**, 170; 1898.

² Hulett, *Physical Review*, **22**, 48; 1906.

parison given above are probably due to the fact that they had not adjusted themselves completely during two days to the new temperature condition.

13. THE ELECTRICAL DETERMINATIONS.¹

After the apparatus was set up, ready for the electrical part of the experiment, the insulation resistance between the two separate windings of the stationary coil with the twisted lead wires was tested and found to be 30 megohms.

The arrangement of the apparatus is given diagrammatically in Fig. 6. From a battery B of 120 volts the current was sent through the standard resistance R and a rheostat W_1 capable of continuous

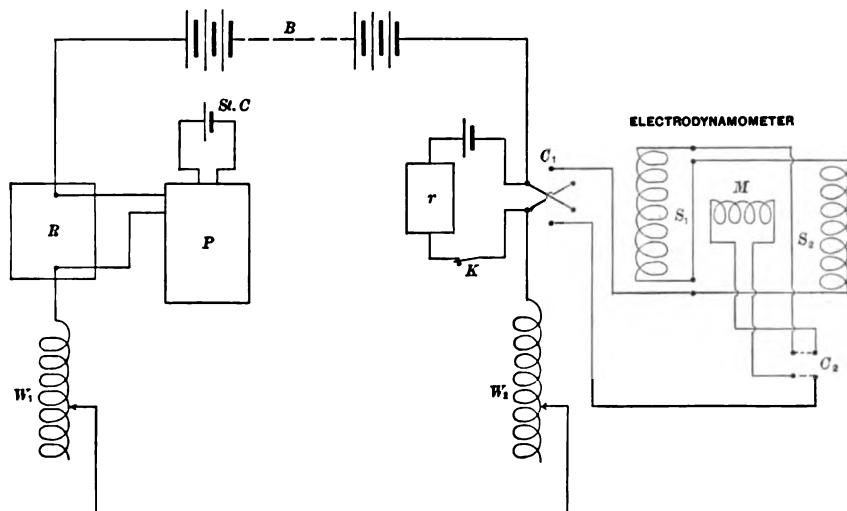


Fig. 6.—Plan of Connections.

adjustment, which could be operated by the observer stationed at the potentiometer P . The latter served for the comparison of the standard cells with the potential difference at the terminals of the standard resistance. From the resistance W_1 the current was led to the electro-dynamometer, passing on its way through a rheostat W_2 , which could be adjusted by small steps and was operated by the observer stationed at the electro-dynamometer. The two commutators allowed a reversal of the current passing through the whole instrument, and for each of the directions in the stationary coil a

¹ I wish to express my indebtedness to Prof. C. M. Jansky and Mr. C. A. Pierce for valuable assistance in the experimental work.

reversal in the movable coil. The four possible combinations of the current were made necessary in order to eliminate (1) the effect of the earth's magnetic field and (2) the influence of the lead wires. The axis of the fixed coil was approximately parallel to the magnetic meridian. The effect of the earth's field on the torque is proportional to the first power of the current, that due to the electro-dynamometer proportional to the second power. The formulæ for the torque due to i_1 and i_2 (the values of the current in the two directions, respectively), this torque being balanced by the torsion of the wire, which is the same in each case, are as follows:

$$T = ai_1^2 + bi_1$$

$$T = ai_2^2 - bi_2$$

or eliminating b

$$T = ai_1i_2$$

Taking the geometrical mean of the two currents actually found will therefore eliminate the influence of the earth's field. If the lead wires have an appreciable influence, the constant a consists of two parts, one, c , due to the effect of the two coils upon each other and the other, d , to the lead wires. By reversing the direction of the current through the two coils with respect to each other, and as before, finding two currents balancing the moment T we obtain finally

$$T = (c + d) i_1 i_2 \text{ and } T = (c - d) i_3 i_4$$

Multiplying these two equations together we have

$$T^2 = (c^2 - d^2) i_1 i_2 i_3 i_4$$

or $T = c \sqrt{i_1 i_2 i_3 i_4} = c I$, where I is the geometrical mean of the four currents, if we neglect the factor $\sqrt{1 - \frac{d^2}{c^2}}$ after taking the square root. Experiments showed that $\frac{d}{c}$ was about $\frac{1}{12000}$ and hence the factor neglected differs from unity by only one

in 288,000,000. Hence, taking the geometrical mean of the four currents as the value of I completely eliminates the effect of the earth's field and of the leads. In all our experiments we twisted the torsion head by an angle nearly 90 degrees and determined the cur-

rent necessary for a balance, then calculated its value for exactly 90 degrees, i. e., the current which would hold the movable coil in its original position, if the torsion head were turned by exactly 90 degrees. The geometric mean of the four values found for the different combinations of the current was then used for further calculation. We selected 90 degrees because in this case the elastic after-effect of the wire is very small for a twist lasting, as in our experiments, only a few minutes. It was, however, taken into account, for immediately after the current was broken, the torsion head was turned back, the movable coil brought nearly to rest and then the small elongations on either side noted for several minutes. From the calculated resting points extrapolation would give us the resting point corresponding to the moment when the coil was turned back. The correction applied hardly ever amounted to more than one millimeter, which with a distance of 257.5 cm of the observer's telescope and scale from the mirror of the movable coil corresponds to a change of only 1 in 8,100. Usually the correction was in the neighborhood of 0.5 mm or 1 in 16,000. The twist of the torsion head was determined by means of the silvered glass cube, described on page 50. It was placed on top of the torsion head in such a position that the side whose adjacent angles were b ($= 89.9994$ degrees) and c ($= 90.0050$ degrees) pointed toward the telescope and the reflection of a scale placed at the same distance from the cube as that of the lower scale from the mirror of the movable coil was observed. A turning to larger numbers meant a turning of the cube through angle b , in the opposite direction through angle c . From the original reading and that after turning, the angle of twist which was always very close to 90 degrees could be accurately calculated.

The procedure for any one of the four observations, necessary for a complete determination was as follows: The temperature was read on the three thermometers distributed along the wire and the reading of the potentiometer with the standard cell No. 813 was taken. One observer was stationed at the electro-dynamometer, another at the potentiometer, and an assistant was ready to turn the torsion head at a given signal. Readings were taken of the scales reflected by the glass cube and the mirror of the movable coil. Then at a noted time the torsion head was turned until the reflection from the

next side of the cube was nearly the same as from the original. The observer at the dynamometer adjusted the rheostat W_2 (see Fig. 6) until the reading of the scale reflected from the mirror of the movable coil was very nearly the same as before, using at the same time the damping device described above, and he called out to the observer at the potentiometer as soon as the coil passed through the original resting point. The assistant at the potentiometer had been following the variations in the current and at the signal given by his companion held the current constant by regulating the rheostat W_1 . This could be done within a few units of the last dial of the potentiometer; that is, within a few parts in 100,000. During this time the first observer read the extreme swings of the movable coil, usually three, then opened the circuit and called out to the assistant to turn the torsion head back to its original position. After this the readings of the swinging coil were taken for several minutes to allow correction for elastic after-effect. Finally the new position of the torsion head was accurately read, and the reading of the potentiometer with the standard cell repeated. It was found necessary to slightly tap the stand on which the two mercury cups leading to the movable coil were placed because it was found that without this precaution the surface tension of the mercury slightly affected the final position of the coil. Several schemes were employed to overcome this trouble but nothing was found to be as efficient as the slight tapping, which was done by a fork passing through the door of the stationary coil. The assistant who turned the torsion head thus kept the mercury agitated throughout the experiment. This shaking was responsible for the difficulty in keeping the current more constant than to the degree mentioned above, but, as we found from several unsuccessful determinations, it could not be dispensed with.

Four such observations served for the determination of the current necessary to balance the torsional moment of the wire. Since the latter was twisted in every case 90 degrees, the torsional moment due to the wire was $\frac{\pi}{2}\tau$. The strength of the field at the center of the stationary coil is

$$H = \frac{4\pi N}{\sqrt{D^2 + L^2}} I = CI$$

Where N is the number of turns on the stationary coil,
 D and L are the diameter and length of the stationary coil,
 I the current expressed in c. g. s. units.
 The effective area of the movable coil is given by

$$A = \pi r^2 n,$$

where n is the number of turns, and r the radius of the coil.
 The torsional moment produced by a current I flowing through the
 electro-dynamometer is then

$$T = ACI^2$$

The values for C and A have been given above, and are

$$C = 165.992 \text{ cm}^{-1},$$

$$A \text{ for the larger movable coil} = 8509.9 \text{ cm}^2,$$

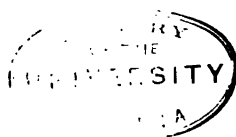
$$A \text{ for the smaller coil} = 3724.3 \text{ cm}^2.$$

The value of the current expressed in c. g. s. units is therefore
 given by

$$I^2 = \frac{\pi \tau}{2AC} = \frac{1}{1412575} \cdot \frac{\pi}{2} \cdot \tau \text{ for the larger coil,}$$

$$= \frac{1}{618203} \cdot \frac{\pi}{2} \cdot \tau \text{ for the smaller coil.}$$

The value for τ to be used with the larger coil was $309.030 \text{ gm cm}^2 \text{ sec}^{-2}$ at 25° , for the smaller coil $308.688 \text{ gm cm}^2 \text{ sec}^{-2}$ at 25° . In each determination this value has to be corrected for temperature. Seven complete determinations were made, three with the smaller coil and four with the larger. The standard cell used as standard of comparison was the Weston cell No. 813. Since the electromotive forces of all the others are accurately known in terms of this, their values in absolute units can easily be calculated. In Tables XV and XVI a summary of these experiments is given. Under (A) we find the results of the four partial observations, corrected for a torsion equal to 90 degrees, the different directions of the current being indicated by a , b , c , and d . The effect of the earth's magnetic field is quite apparent; on the other hand the effect of the leading-in wires is very small. Under (B) I give a short outline of the final calculation. In the first line we find the geometrical mean of the



four readings of the potentiometer given under (A), in the third the torsional moment for unit angle of twist corrected for the temperature of the dynamometer. In the next line follows the value of the current in c. g. s. units, calculated as shown above, and corrected for the magnetic influence of the bracket, then the potential difference

TABLE XV.

Determinations with Small Movable Coil.

(A)

Direction of Current	Reading of Potentiometer		
	Experiment 1	Experiment 2	Experiment 3
a	0.82861	0.82875	0.82844
b	.85145	.85172	.85181
c	.82881	.82868	.82883
d	.85173	.85176	.85163

(B)

Pot. Diff. observed.....	0.84008	0.84017	0.84010
Temperature.....	25°20	25°27	25°18
Modulus of torsion.....	308.736	308.727	308.738
Current in c. g. s. units.....	0.0280029	0.0280025	0.0280030
Pot. Diff. calculated.....	0.840143	0.840131	0.840146
Pot. Diff. <u>calculated</u> observed.....	1.00008	0.99995	1.00005
Emf. of Std. Cell No. 813.....	1.01890	1.01877	1.01887

calculated from Ohm's law. In the experiments with the smaller coil the standard resistance was 3.00020 ohms at 25°, i. e., the temperature of the well stirred oil bath in which they were kept; with the larger coil the resistance was 6.00038 ohms at 23°. These resistances were accurately compared with the standards of the Bureau. In the remaining two lines of the tables are given the ratio between the calculated and the observed potential differences and the emf. of our reference cell Weston No. 813 at 21°, as determined by the absolute measurement.

TABLE XVI.

Determinations with Large Movable Coil.

(A)

Direction of Current	Reading of Potentiometer			
	Experiment 4	Experiment 5	Experiment 6	Experiment 7
a	1.08971	1.08971	1.08959	1.08958
b	1.13573	1.13561	1.13579	1.13571
c	1.08956	1.08970	1.08966	1.08951
d	1.13554	1.13546	1.13536	1.13579

(B)

Pot. Diff. observed.....	1.11240	1.11238	1.11237	1.11241
Temperature	23°44	23°62	23°47	23°60
Modulus of torsion	309.200	309.180	309.195	309.182
Current in c. g. s. units.	0.0185394	0.0185388	0.0185393	0.0185388
Pot. Diff. calculated...	1.11243	1.11240	1.11242	1.11240
Pot. Diff. ^{calculated} observed.....	1.00003	1.00002	1.00004	1.00001
Emf. of Std. Cell No. 813	1.01885	1.01884	1.01886	1.01883

Taking the average of the seven determinations of the emf. of Weston cell No. 813 at 21°, giving the first three only half the weight of the last four, we obtain:

$E = 1.01884$ volts, only 2 parts in 100,000 higher than its certified value.

TABLE XVII.

Electromotive Forces of Clark and Weston Cells, as found by Absolute Determinations.

Type of Cell.	25° C	20°	15°
F Series (electrolytic)	1.01827	1.01847	1.43296
E, K and O Series (electrolytic)	1.01833	1.01853	
C Series (chemical).....	1.01857	1.01877	
R Series (electrolytic)	1.42040		

From this the electromotive forces of the various standard cells compared with the Weston as shown above can be easily found, and are given in Table XVII.

The value found by the Reichsanstalt for the chemically prepared cadmium cells at 20° is 1.0186 volts or 0.00017 volt lower.

ELECTROCHEMICAL EQUIVALENT OF SILVER.

We are now able to calculate the electrochemical equivalent of silver in absolute measure. In a former work¹ I have determined its value in terms of a Weston cell (No. 804 a B. S.). The emf. of this cell was assumed at that time to be 1.01954 volts at 20°; taking the emf. of a reference Clark as 1.434 volts at 15°. By direct comparison with the reference standard No. 813 used in this investigation its emf. was found to be 0.00012 volt lower than No. 813 or 1.01870 volts at 20°. Its emf. at 20° as found by the present absolute determination is therefore 1.01872 volts. The relative value of the electrochemical equivalent found before was 1.11683 mg per coulomb, and we obtain therefore as the electrochemical equivalent of silver, when a porous cup coulometer is used,

1.11773 mg per coulomb,

a result agreeing well with former determinations, except the French (see Table I).

It should be mentioned that it had been my intention to use also a bifilar suspension in this work and that some experiments in this direction had been made, but could not be brought to a conclusion on account of lack of time. The bifilar suspension has the advantage of being free from elastic after-effect, but the calculation of the torsional moment (without the use of a moment of inertia) presents great difficulties, since a slight change in relative tension of the two wires is accompanied by considerable change in the torsional moment. I was also prevented from trying finer suspensions or spiral springs, which might be used by having a part of the load carried by wires which take no part in the torsion. The results obtained in the present investigation are, however, highly satisfactory in showing an agreement among themselves hardly expected, especially between the two different movable coils.

¹ Guthe, this Bulletin, 1, 21; 1904.

THE GRAY ABSOLUTE ELECTRODYNAMOMETER.

By Edward B. Rosa.

1. SIMPLE THEORY OF THE INSTRUMENT.

This instrument consists of a cylindrical coil of wire of a single layer suspended within a larger fixed coil, also cylindrical and of a single layer, the ratio of the radius of each coil to its length being as one to the square root of 3. When these two coils have this particular shape and their centers coincide, the expression for the torque which in general is given by a series of terms is simplified by the disappearance of all terms after the first up to the seventh, and this and succeeding terms are usually small enough to be neglected.¹ The torque is then expressed, if the axes are at right angles, as is usually the case in practice, by the following extremely simple formula:

$$T = \frac{2\pi^2 r^2 N_1 N_2 I_1 I_2}{c} \quad (1)$$

where r is the radius of the suspended coil.

N_1 and N_2 are the whole number of turns of wire on the fixed and suspended coils respectively,

I_1 and I_2 are the currents in the two coils respectively, which may or may not be the same current,

c is half the diagonal of the fixed coil $= \sqrt{a^2 + b^2}$, a being the radius of the fixed coil and $2b$ its length.

As stated above $\frac{a}{2b} = \frac{1}{\sqrt{3}}$.

¹ Gray, *Absolute Measurements*, Vol. II, part I, p. 276. Patterson, *Physical Review*, 20, 1905.

The seventh and succeeding terms, which amount only to extremely small correction terms, are appreciable only when the suspended coil is relatively large; they can readily be calculated if necessary.

The formula may also be written as follows:

$$\left. \begin{aligned} T &= \frac{2\pi N_1 I_1}{c} \cdot \pi r^2 N_2 I_2 \\ &= H I_1 \times A I_2 \end{aligned} \right\} \quad (2)$$

where $H = \frac{2\pi N_1}{c}$, and $A = \pi r^2 N_2$. H is the magnetic force at the

center of the fixed coil, due to unit current flowing through the fixed coil, and A is the sum of the areas of the several turns of the suspended coil. Thus the torque is the same as though the suspended coil were hung in a uniform field of strength $H_1 = H I_1$. The field is, of course, not uniform, the center O being a maximum point with respect to the axis $A B$ of the fixed coil (Fig. 1), and a minimum point with respect to the axis of the suspended coil, which is a diameter of the fixed coil. This saddle-shaped field gives a torque, however, exactly the same (within 1/27,000 when $r = 1/2 a$ and still closer for smaller values of r such as are used in practice), as though the field were uniform and of the same intensity as it has at the center, O .

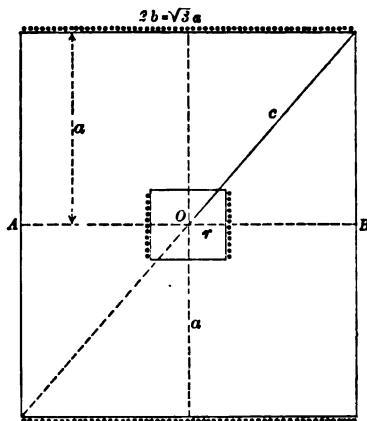


Fig. 1.

This form of electro-dynamometer has been used by Patterson and Guthe¹ and Carhart and Guthe² in the absolute measurement of current and of the electromotive force of a standard cell, and has been used by Guthe in a recent very careful determination at the Bureau of Standards, an account of which is published in the present number of this Bulletin. The instrument is extremely well

¹ Physical Review, 7; 1898.

² Physical Review, 9; 1899.

adapted to precision measurements, the quantity most difficult to determine with sufficient accuracy being the torque. Guthe and others have, however, succeeded so well in this respect that it seems worth while to construct yet another instrument of the same style, and to refine the construction and measurements still further if possible, with the hope of fixing yet more definitely the value of the standard cell.

I propose to examine the theory of the instrument with respect to sources of error arising in its construction and measurement, leaving aside for the present the question of errors arising in carrying out the experiment.

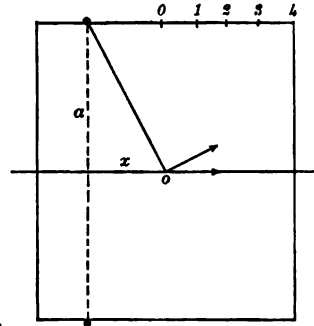


Fig. 2.

The formula (2) expresses the torque in terms of the field H at the center of the fixed coil. The force at O in the direction of the axis due to unit current in a single turn of wire is

$$H_1 = 2\pi \frac{a^2}{(a^2 + x^2)^{3/2}} \quad (3)$$

To find the force due to a current sheet of length $2b$ with a current of n units per centimeter, we integrate along the length of the current sheet, thus

$$H = 2\pi a^2 \int \frac{ndx}{(a^2 + x^2)^{3/2}} = 2\pi n \left[\frac{x}{\sqrt{a^2 + x^2}} \right]_{-b}^{+b} = 2\pi n \frac{2b}{\sqrt{a^2 + b^2}} \quad (4)$$

If the current sheet is replaced by a winding of fine wire having n turns per centimeter and carrying unit current, there being altogether $N_1 = 2bn$ turns, we have

$$H = \frac{2\pi N_1}{\sqrt{a^2 + b^2}} = \frac{2\pi N_1}{c} \text{ as above, (2),}$$

provided the single layer winding of wire has the same magnetic effect at the center that the current sheet has. In other words, the formula given above for the Gray dynamometer applies strictly to the case of a current sheet on each of the two cylindrical coils, and

we have to inquire whether a winding of insulated wire is exactly equivalent to the current sheet, and how to measure the coils in getting a , c , and r for the formula.

**2. A SINGLE LAYER OF WIRE EQUIVALENT TO A CURRENT SHEET.
LENGTH OF THE COILS.**

Considering first the case of the fixed coil, it is evident that if it be wound with flat strip of negligible thickness and of width equal to the outside diameter of the insulated wire, and the same number of turns are applied as there are of wire, then the edges of the strip will meet (without making contact, it is supposed) and the current flowing through the strip will envelop the cylinder like a current sheet. If there are n turns per centimeter, unit current in the strip will have the same magnetic effect at the center of the coil as a current of n units per centimeter, in the current sheet. We may therefore consider the formula exact for the assumed winding of the strip, and consider whether the winding of insulated wire is magnetically equivalent to the strip. In the fixed coil of the electro-dynamometer used by Guthe there are about 20 turns per centimeter. I shall therefore consider the case of wire .05 cm in diameter. Equation (4) gives the force at the center due to any portion of a current sheet, and if we insert for the limits the values of x at the two edges of a single turn of the strip we shall have the force due to one turn of strip. Thus, since $n=20$ and $a=25$ cm, we have for a single turn at the center of the cylinder, the values of x for the edges being $+.025$ and $-.025$,

$$H_1 = 2\pi \times 20 \left[\frac{x}{\sqrt{a^2 + x^2}} \right]_{-.025}^{+.025} = \frac{2\pi}{a} \left[\frac{0.5}{\sqrt{1 + (.001)^2}} + \frac{0.5}{\sqrt{1 + (.001)^2}} \right] \quad (5)$$

$$\text{or } H_1 = \frac{2\pi}{a} \times .9999995$$

A single turn of infinitely fine wire at the center ($x=0$) would give a force H at the center O equal to

$$H_s = 2\pi \frac{a^2}{(a^2)^{\frac{3}{2}}} = \frac{2\pi}{a}$$

The difference $H_s - H_1$ is only one part in 2,000,000 of the force, and hence the single filament is practically equivalent to the strip.

If the strip were 1 cm wide the difference would be 400 times as great, or one part in 5,000. If the strip is wound on edge and carries unit current the force due to an element dy of one turn at a distance $a+y$ (where a is measured to the center of the strip) is

$$dH_1 = \frac{2\pi}{a+y} \cdot \frac{dy}{2a} \cdot \frac{a}{a+y}$$

where $2a$ is the depth of the strip, and $\frac{a}{a+y}$ is the relative current density at the element dy as compared with its value at the center of the strip. This is not strictly the mean value, but it is nearly enough true for our purpose when the radius is large.

The whole force is then

$$H_1 = \frac{2\pi a}{2a} \int_{-a}^{+a} \frac{dy}{(a+y)^2} = \frac{2\pi a}{2a} \left[-\frac{1}{a+y} \right]_{-a}^{+a} \quad (6)$$

$$= \frac{2\pi}{a} \left[\frac{a^2}{a^2 - a^2} \right]$$

If as before, $a = 25$ and $a = .025$,

$$H_1 = \frac{2\pi}{a} \left[1.000001 \right]$$

$$\text{whereas } H_2 = \frac{2\pi}{a}$$

$$\text{Thus } H_1 - H_2 = \frac{2\pi}{a} \times 0.000001$$

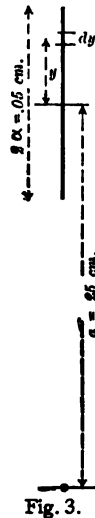


Fig. 3.

That is, the force due to unit current in the strip on edge exceeds that due to unit current in a filament at its center by one part in a million. If the strip were 1 cm deep the difference would be appreciable but small.

It is thus evident that for a wire of small diameter at the center of the cylinder the effect of the breadth of the wire is to make the force at the center slightly less, and the effect of the depth is to make it slightly greater, than if the current were concentrated at the center of the cross section of the wire; hence the resultant difference between the effect of unit current in a round wire .05 cm in diameter and a filament at its center is even less than that between the strip

and the filament, and hence the wire may be considered as equivalent to the strip to within less than one in a million.

If the strip is near the end of the cylinder where $x=20$, say, its force at the center is found similarly from equation (4). Thus,

$$H_1 = 2\pi \times 20 \left[\frac{x}{\sqrt{a^2 + x^2}} \right]_{20-a}^{20+a} \quad (7)$$

$$= 2\pi \times 20 \left[\frac{20.025}{\sqrt{625 + (20.025)^2}} - \frac{19.975}{\sqrt{625 + (19.975)^2}} \right] = .11966704$$

whereas the force due to a filament at the center of the strip is

$$H_2 = 2\pi \frac{625}{(625 + 400)^{\frac{3}{2}}} = .11966695$$

Thus, H_1 is larger than H_2 by about one part in a million, whereas at the center of the cylinder it is smaller by one part in 2,000,000. At a distance of 12.5 cm from the center the two are exactly equal. Thus the difference between the winding of fine insulated wire of 20 turns per cm and thin strip which is exactly equivalent to a current sheet is probably less than one part in 5,000,000. If the strip is 1 cm wide instead of one-twentieth of a centimeter, the difference at the end of the cylinder is only one part in 8,600, the effect of the strip being greater, whereas at the center of the cylinder the difference, as we saw above, was one part in 5,000, the effect of the strip being less. Thus, the average difference will probably be less than one in 12,500 making an error in the current of only one in 25,000. In other words, if a winding of strip 1 cm wide for which the formula of the electro-dynamometer applies were replaced by a winding of fine wire 1 cm apart, the error would scarcely be appreciable. It is to be noted that the length of the cylinder to be used in the formula would be the total length of the equivalent current sheet, and not the length of the winding of wire. Thus if the cylinder were 43 cm long, the 43 turns of strip would be replaced by 43 turns of fine wire. The distance between the first and last wires would be 42 cm; but the length of the cylinder would be taken as 43 cm, and not 42, in calculating the force at the center. This is an extreme case, but the same principle applies always, namely, that the length of the

cylinder is to be the length of the equivalent current sheet, and in a winding of insulated wire this is the *overall length of the winding*, AB, *including the insulation on the first and last wires*. It is generally supposed that the measurement should be made a little short of the outside dimensions as *ab*, not including the insulation. The difference is not large in a winding of fine wire, but it is appreciable even in that case; amounting to about two parts in 10,000 in a winding of 20 turns per centimeter, with a radius of 25 cm.

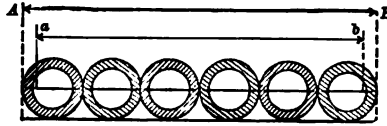


Fig. 4.

3. EFFECT OF IRREGULAR WINDING.

Equation (3) gives the effect at the center of unit current in a single turn at distance x from the center. Differentiating this with respect to x we have

$$dH = -2\pi a^2 \frac{3xdx}{(a^2 + x^2)^{\frac{5}{2}}} \quad (8)$$

which gives the effect of displacing the wire parallel to itself by a distance dx . Dividing (8) by (3) we have for the relative change in the magnetic force due to a displacement dx ,

$$\frac{dH}{H} = -\frac{3xdx}{a^2 + x^2}$$

Thus the relative change is proportional to x , the distance from the central plane and inversely proportional to the square of the distance of the wire from the center of the cylinder O. To find the actual value of this change we may substitute in (9) numerical values for a and x . Putting $a = 25$ cm and assuming a displacement dx of only 0.1 mm, and giving x values corresponding to the points 0, 1, 2, 3, 4, in Fig. 2 (that is at the center, and at quarter, half, and three-quarters of the distance from the center to the end, and finally at the end), we have

$$\text{At point 4, } x = \frac{1}{2}\sqrt{3} a, \quad \frac{dH}{H} = \frac{1}{1700}$$

$$\text{At point 3, } x = \frac{3}{8}\sqrt{3}a, \frac{dH}{H} = \frac{1}{1820}$$

$$\text{At point 2, } x = \frac{1}{4}\sqrt{3}a, \frac{dH}{H} = \frac{1}{2300}$$

$$\text{At point 1, } x = \frac{1}{8}\sqrt{3}a, \frac{dH}{H} = \frac{1}{4000}$$

$$\text{At point 0, } x = 0, \frac{dH}{H} = \frac{1}{\infty}$$

Thus we see that the effect of displacing a given wire by so little as 0.1 mm produces an appreciable effect, and if a large number of wires are so displaced, due to irregular winding, the effect may be serious if it is not calculated and a correction applied. Of course, in any irregular winding the displacements will be both plus and minus, and hence the resultant effect will be the difference due to the two kinds of displacement. The effect is, however, so large that this difference may be very important in precise work.

A good approximation to a correction for the effect of irregular winding may be made by dividing the winding into a number of sections and calculating the magnetic effect at the center due to each section by equation (4) using for the limits the values of x at the edges of the section. This requires a careful calibration of the coil, such as Guthe has made in his recent work. He measured the length of each 50 turns of the winding at several different equidistant elements of the cylinder, and took the averages as the breadth of each section. There being 872 turns in all, there are 17 sections of 50 turns each and 1 section of 22 turns. The breadths of these sections are given in column 3 of Table 1. Column 4 gives the distance from one end of the successive boundaries, found by adding successively the numbers in column 3. Column 5 gives the distance from the center of each boundary, found by subtracting the half length b from the numbers of column 4. These are the values of x to be used in equation (4) in calculating the successive values of H at the center. Column 6 gives the corresponding values of $\frac{H}{2\pi}$.

The magnetic force due to unit current in the first section is found as follows:

$$H_1 = 2\pi n_1 \left[\frac{x}{\sqrt{a^2 + x^2}} \right]_{x_1}^{x_2} \quad (10)$$

where n_1 is the number of turns per centimeter, and x_1 and x_2 are the distances of the edges of the section of n turns from the center.

Since $x_1 - x_2$ is the breadth of the section, $n_1 = \frac{n}{x_1 - x_2}$. Hence, equation (10) becomes

$$H_1 = \frac{2\pi n}{x_1 - x_2} \left[\frac{x_1}{\sqrt{a^2 + x_1^2}} - \frac{x_2}{\sqrt{a^2 + x_2^2}} \right] \quad (11)$$

Substituting $n = 50$,

$$x_1 = 21.6382,$$

$$x_2 = 19.1437,$$

$a = 24.9812$, the mean radius of the windings, we find that $H_1 = 2\pi \times 0.931245$. The same operation being carried out for each section gives the values contained in the last column of Table I, from which it follows that the magnetic force at the center of the coil is $H = 165.992$.

If we assume the winding uniform we should have, since c the half diagonal $= \sqrt{a^2 + b^2} = \sqrt{1092.272}$,

$$H = 2\pi \frac{872}{\sqrt{1092.272}} = 165.778.$$

This value of H , the magnetic force at the center of the cylinder, is too small by 0.214, or 1 in 800; and this would cause an error of 1 in 1,600 in the value of the current found. It is thus seen that the error which would arise from assuming the winding uniform in this case amounts to more than the uncertainty in the absolute value of the standard cell. As this coil was carefully wound, and its winding looks very smooth and uniform, there is no reason to suppose it is worse than such coils often are. Hence, we may conclude that any value obtained from an absolute dynamometer of this type in which uniformity of winding is not assured, or in which the effect of irregularity in the winding is not taken into

account, is untrustworthy. It is of course desirable that the winding be put on as uniformly as possible, so that the calculated correction may be quite small.

TABLE I.

Showing the Breadth and Location of Each Section of Winding of Guthe's Electrodynamometer, and Its Magnetic Force at the Center.

Number of section	Number of turns	Breadth of section	Distance from end	Distance from center = x	Values of $H + 2\pi$ for each section
		cm	0.0000	-21.6382	
1	50	2.4945	2.4945	-19.1437	0.931245
2	50	2.4940	4.9885	-16.6497	1.075830
3	50	2.4970	7.4855	-14.1527	1.234935
4	50	2.4887	9.9742	-11.6640	1.403505
5	50	2.4887	12.4629	- 9.1753	1.573075
6	50	2.4758	14.9387	- 6.6995	1.731570
7	50	2.4652	17.4039	- 4.2343	1.864215
8	50	2.4709	19.8748	- 1.7634	1.956820
9	50	2.4696	22.3444	+ 0.7062	1.997730
10	50	2.4639	24.8083	+ 3.1701	1.981245
11	50	2.4575	27.2658	+ 5.6276	1.909995
12	50	2.4588	29.7246	+ 8.0864	1.793570
13	50	2.4766	32.2012	+10.5630	1.644875
14	50	2.4804	34.6816	+13.0434	1.479280
15	50	2.4978	37.1794	+15.5412	1.309140
16	50	2.4953	39.6747	+18.0365	1.144890
17	50	2.4946	42.1693	+20.5311	0.993455
18	22	1.1071	43.2764	+21.6382	0.393080
		43.2764		Total ..	26.418455

$$H = 2\pi \times 26.418455 = 165.992$$

The winding of wire being spiral, it is evident that the distance from the central plane to the end turn is different at different elements of the cylinder. One half a turn will be farther and the other half nearer than the mean distance, the average displacement being one-fourth of the diameter of the wire. The resultant will be the difference of the effects of these two opposite displacements, and

this compared with the total force at the center will be extremely small. Substituting the numerical values in equation (9) it can readily be shown that the variation due to the spiral winding from that calculated on the supposition that each turn is in a plane perpendicular to the axis is not as much as one part in two million.

4. EFFECT OF AN ERROR IN THE RADIUS OR LENGTH OF THE FIXED COIL. TEMPERATURE COEFFICIENTS.

To find the effect on the computed value of the magnetic field due to a small error in measuring the radius we differentiate the expression for the total force at the center.

Since

$$H = \frac{2\pi N}{(a^2 + b^2)^{\frac{3}{2}}},$$

$$dH = -2\pi N \frac{ada}{(a^2 + b^2)^{\frac{3}{2}}}$$

$$\frac{dH}{H} = -\frac{ada}{a^2 + b^2} = -\frac{a^2}{a^2 + b^2} \cdot \frac{da}{a} = -\frac{4}{7} \frac{da}{a} \quad (12)$$

since $b^2 = \frac{4}{3}a^2$. Similarly,

$$\frac{dH}{H} = -\frac{bdb}{a^2 + b^2} = -\frac{b^2}{a^2 + b^2} \cdot \frac{db}{b} = -\frac{3}{7} \frac{db}{b} \quad (13)$$

Thus, the relative error in the magnetic field is four-sevenths of the relative error in the radius, and three-sevenths of the relative error in the measurement of the length. If an error of 0.1 mm is made in determining the mean diameter, which is 50 cm, that is one part in 5,000; the error in the computed magnetic field will be 1 part in 8,750. If an error of 0.1 mm be made in measuring the length of the winding, which is 43.3 cm, that is one part in 4,330; the error in the computed magnetic field will be 1 in 10,100.

If a and b change together due to change of temperature, the relative change in the magnetic field due to unit current will be the sum of the effects due to change in a and b , or

$$\frac{dH}{H} = -\left(\frac{3}{7} + \frac{4}{7}\right) \frac{da}{a} = -\frac{da}{a} \quad (14)$$

That is, the temperature coefficient for the magnetic field is negative, and equal numerically to the temperature coefficient for the material on which the coil is wound. If the cylinder is marble, this is 1 part in 100,000 per degree Centigrade. For plaster of Paris Guthe found it to be 1 part in 40,000 per degree.

This expression for the temperature coefficient may be directly deduced as follows. If the coil expands equally in all directions, thus maintaining the same form, $b^3 = \frac{3}{4}a^3$ always, and

$$H = \frac{2\pi N}{\left(a^2 + \frac{3}{4}a^2\right)^{\frac{1}{2}}} = \frac{2\pi N}{\left(\frac{7}{4}\right)^{\frac{1}{2}}} \frac{1}{a}$$

$$dH = - \frac{2\pi N}{\left(\frac{7}{4}\right)^{\frac{1}{2}}} \cdot \frac{da}{a^2}$$

$$\therefore \frac{dH}{H} = - \frac{da}{a}, \text{ as above.}$$

5. EFFECT OF THE OPENING IN THE COIL FOR THE SUSPENSION.

An opening is left in the winding of the fixed coil through which the suspension passes. Several wires near the middle plane are carried around the oblong hard-rubber bushing which lines this hole, the length of the bushing being perhaps 2 cm. To calculate the effect of this irregularity in the winding we may consider the displacement laterally of four wires through a distance of 2 mm on each side of the central plane, and a simultaneous radial displacement of the same wires through half a millimeter, as they are laid upon the adjacent wires. The length of wire so displaced may be taken as 3 cm. These dimensions are greater than those of Guthe's dynamometer, and will give an outside value for the correction.

Equation (9) gives the relative change in the magnetic field at the center due to a displacement of any turn of wire parallel to itself through a distance dx . Integrating it with respect to x we have the relative change as a wire is displaced through a (small) finite distance $x_2 - x_1$. For a displacement from $x_1 = 0$ to $x_2 = 0.2$ cm we thus have ($a = 25$ cm),

$$\int_0^{0.2} \frac{3x dx}{a^2 + x^2} = \frac{3}{2} \log \frac{a^2 + 0.04}{a^2} = \frac{3}{2} \cdot \frac{0.04}{625} = \frac{1}{10,000} \text{ approximately.}$$

Displacing the other three wires from $x_1 = .05$ to $x_2 = .25$, $x_1 = .10$ to $x_2 = .30$, $x_1 = .15$ to $x_2 = .35$ respectively, produces a slightly greater change, the mean for the four being 1 part in 6,000. This supposes the wires displaced for the whole length of each turn, whereas the displacement is only for 3 cm out of 157. Hence the total relative change is only one part in 300,000 of the eight central wires, or one part in about one hundred and fifty million of the whole force at the center.

Similarly, by the use of equation (12) we can estimate the effect of half a millimeter radial displacement of the same eight wires, four

on each side of the central plane. For $da = .05$ cm, $\frac{da}{a} = .002$. Hence for 3 cm of the 157 in one circumference we have

$$\frac{dH}{H} = \frac{4}{7} \times .002 \times \frac{3}{157} = 0.000022.$$

But these eight wires exert only about one-fiftieth of the total force at the center, hence the relative change in the total force due to the radial displacement is less than one in two million. Hence we see that the change in the field due to the irregularity in the winding occasioned by this opening is so small that it is not wise to cramp the experiment by making it unduly small.

6. EFFECT OF ERROR IN MEASURING R, AND EFFECT OF THICKNESS OF THE WIRE OR STRIP.

Since $A = \pi r^2 N$, the torque is proportional to the square of the radius. Thus

$$\frac{dT}{T} = 2 \frac{dr}{r} \quad (15)$$

That is, the relative error in the computed torque is twice the relative error in measuring the radius of the small coil. It is therefore important to measure this radius with great exactness, and one

should use a moving coil large enough to permit this measurement to be made with the necessary accuracy. This error includes, of course, not simply the error in measuring the cylinder, but of getting the exterior diameter after winding. Flat strip is better than round wire on this account. The temperature coefficient of the torque, due to expansion or contraction of the suspended coil, is, of course, twice the temperature coefficient of the material of which the suspended cylinder is made. It is usual to take the radius of the coil as the distance from the center of the cylinder to the center of the wire or strip; or, what is the same thing, the mean of the radius of the bare cylinder and of the cylinder after it is wound. This is not quite exact, whether we assume the current to be uniformly distributed through the cross section of the wire, or assume it to be inversely proportional to the distance from the center, but for thin strip the error is inappreciable in either case, as the following demonstration shows. The torque is

$$T = CI_1\pi r_0^2 N_2 I_2,$$

where r_0 is the equivalent radius of the suspended coil and I_2 is the total current flowing in it.

First, assume that the current density at any point in the strip is inversely proportional to its distance r from the axis of the cylinder, then

$$di = a \frac{dr}{r}$$

The whole current I in the strip is

$$I = a \int_{r_1}^{r_2} \frac{dr}{r} = a \log \frac{r_2}{r_1}, \therefore a = \frac{I}{\log \frac{r_2}{r_1}}.$$

If r_0 is the equivalent radius of the strip, that is, the radius of the equivalent current sheet,

$$r_0^2 I = a \int_{r_1}^{r_2} r^2 di = \frac{I}{\log \frac{r_2}{r_1}} \int_{r_1}^{r_2} r dr = \frac{I}{\log \frac{r_2}{r_1}} \cdot \frac{r_2^2 - r_1^2}{2}$$

$$\text{or, } r_0^3 = \frac{1}{2} \cdot \frac{r_2^3 - r_1^3}{\log \frac{r_2}{r_1}} \quad (16)$$

This expression may be put into more convenient form for calculation as follows, expanding the $\log \frac{r_2}{r_1}$,

$$\begin{aligned} r_0^3 &= \frac{r_2 + r_1}{2} \left(\frac{r_2 - r_1}{\frac{r_2}{r_1} - \frac{r_1}{r_2}} + \frac{1}{3} \left(\frac{r_2 - r_1}{\frac{r_2}{r_1}} \right)^3 - \frac{1}{4} \left(\frac{r_2 - r_1}{\frac{r_2}{r_1}} \right)^4 + \dots \right) \\ &= \frac{r_2 + r_1}{2} \left(\frac{r_1}{1 - \frac{1}{2} \left(\frac{r_2 - r_1}{r_1} \right) + \frac{1}{3} \left(\frac{r_2 - r_1}{r_1} \right)^2 - \frac{1}{4} \left(\frac{r_2 - r_1}{r_1} \right)^3 + \dots} \right) \\ &= \left(\frac{r_2 + r_1}{2} \right) r_1 \left\{ 1 + \frac{r_2 - r_1}{2r_1} - \frac{1}{12} \left(\frac{r_2 - r_1}{r_1} \right)^2 + \frac{1}{24} \left(\frac{r_2 - r_1}{r_1} \right)^3 + \dots \right\} \end{aligned}$$

or, finally,

$$r_0^3 = \left(\frac{r_2 + r_1}{2} \right)^3 \left\{ 1 - \frac{1}{6} \frac{(r_2 - r_1)^2}{r_1(r_1 + r_2)} + \frac{1}{12} \frac{(r_2 - r_1)^3}{r_1^2(r_1 + r_2)} - \dots \right\} \quad (17)$$

Thus the equivalent radius r_0 is less than the mean radius $\frac{r_2 + r_1}{2}$, the difference being chiefly due to the first of the correction terms in the above expression. If $r_1 = 5$ cm, $r_2 - r_1 = .05$ cm, the quantity in brackets above becomes

$$1 - 0.000008292 + 0.000000042 - \dots$$

Thus the correction to be applied to $\left(\frac{r_1 + r_2}{2} \right)^3$ to give the true value of r_0^3 amounts to one part in about 121,000 and is *negative*.

Second, suppose the current density is uniform over the cross section of the strip, then

$$\begin{aligned} di &= \frac{I}{r_2 - r_1} dr \\ r_0^3 I &= \frac{I}{r_2 - r_1} \int_{r_1}^{r_2} r^3 dr = \frac{I}{r_2 - r_1} \left[\frac{r_2^4 - r_1^4}{4} \right] \\ \therefore r_0^3 &= \frac{1}{3} (r_1^3 + r_1 r_2 + r_2^3) = \left(\frac{r_2 + r_1}{2} \right)^3 \left\{ 1 + \frac{1}{3} \left(\frac{r_2 - r_1}{r_2 + r_1} \right)^2 \right\} \quad (18) \end{aligned}$$

Thus the equivalent radius r_0 is again expressed in terms of the mean radius, there being a single correction term which is almost identical with the principal correction term in (17), but is *positive*. Using the same numerical values as before we find the correction amounts to one part in 121,203. Thus the mean radius $\frac{r_2 + r_1}{2}$

is too large by the first hypothesis as to the distribution of current and too small by the second, the error in either case being inappreciable if the thickness of the strip is not more than one per cent of the radius. As we can not be sure of the exact distribution of current, owing to lack of perfect homogeneity in the wire, the better way is to use strip thin enough to make small variations in the distribution unimportant.

We have assumed throughout the foregoing discussion that the changes in the torque due to any irregularity of winding or other departure from the conditions for which the simple formula holds are proportional to the change produced in the field at the center of the cylinder. This is justifiable, since such changes are assumed to be small, and variations could at most be only small quantities of the second order.

The above examination shows that it is of first importance to secure a uniform winding of the fixed coil, and to apply a correction for any unavoidable irregularity that may be found by careful calibration to remain. It also gives the length of the current sheet equivalent to a winding of insulated wire, indicates the order of magnitude of certain possible sources of error, and suggests how to render them inappreciable. These principles will be applied in the new absolute dynamometer, in which a marble cylinder will be used for the fixed coil, shortly to be constructed at the Bureau of Standards.

CONSTRUCTION AND CALCULATION OF ABSOLUTE STANDARDS OF INDUCTANCE.

By J. G. Coffin.

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PART I.

CONSTRUCTION AND MEASUREMENT OF THE STANDARDS.

1. INTRODUCTION.

The work of this paper was undertaken with a view to the construction and calculation of a primary standard of self-inductance for the Bureau of Standards and of one for the Physical Laboratory of

Clark University, Worcester, Mass. It is hoped that the following account of the theory and methods of construction will direct the way and materially decrease the labor of any future undertakings of a similar character. Very little is to be found anywhere on the precautions to be taken in very precise work of this kind, and there were but few formulæ accurate enough for the calculation of the completed standard in absolute measure. It is hoped that the present paper will to some extent fill this lacuna in the literature of the subject.

The construction of a standard of inductance involves many considerations. A primary standard need not necessarily be of a convenient size or weight for ordinary use, but on the other hand it should not be too difficult to manage under favorable circumstances. A standard of any kind should have as nearly as possible the characteristics and properties which are required of standards of length and mass—i. e., as nearly as possible unalterability with age, convenient electrical magnitude for practical measurement, and furthermore all dimensions should be readily determinable to the required degree of accuracy and redeterminable at any time. A convenience which is desirable, but not always attained, is the facility of comparing the standard with one or more of its own subdivisions, thus checking calculated values in terms of itself. This has been made possible in the present instance, as will be explained below.

There are several materials available for forms for standard coils, each with certain advantages and disadvantages. Metal, wood, glass, plaster of Paris, and other compositions were all considered.

Wood has been used for standard inductances and has the advantage of lightness and cheapness, but has also the absolutely prohibitive fault of being liable to warp, however well seasoned and well constructed. Metals must be barred out because of eddy currents which render the value of the inductance a function of the periodicity. Metal forms are also hard to obtain without strains and are very liable to suffer distortion while being worked. Glass and plaster of Paris are excellent materials for coils of small size. Plaster of Paris soaked in paraffin has been tried and found to answer very well, but is more liable to injury than marble, is less rigid, and can not be ground as true. For large coils, glass would be very expensive, and although cost should not be the determining factor in the construction of

primary standards it nevertheless must be considered. Glass in such large masses would also be very difficult to obtain. The material chosen for this coil was marble.

2. THE ADVANTAGES OF MARBLE.

The advantages of marble are manifest. It is comparatively inexpensive and easy to obtain in almost any shape or size. It is unalterable and not attacked by moisture or air, as shown by the marvelous preservation of temples and statues of ancient Greece and Rome, which have stood the test of centuries both underground and exposed to the elements. Its electrical resistance is very high, some samples having a value as high as that of hard rubber. In ordinary work it is feasible to wind bare wire directly on the marble. The coefficient of magnetic susceptibility of marble is very small, and is negative for all kinds investigated; in other words, marble is diamagnetic. The work of A. P. Wills¹ shows that almost any variety of marble can be relied upon to be free from iron and to have a magnetic susceptibility of about -0.8×10^{-6} .

The following values are taken from Wills's paper:

Kind of Marble	Susceptibility
Italian, ordinary	$-.940 \times 10^{-6}$
Italian, statuary I	$-.832 \times 10^{-6}$
Rutland (Vermont)	$-.795 \times 10^{-6}$
Grey, Knoxville	$-.603 \times 10^{-6}$
Italian, statuary II	$-.811 \times 10^{-6}$
Glass	$-.578 \times 10^{-6}$

To avoid any uncertainty, it was thought best to determine the coefficient of susceptibility of samples of the marble from which the cylinders had been cut. Pieces were taken from the cores of the cylinders and their susceptibility measured by the method of A. P. Wills, the work being done by Mr. G. E. Stebbins, a fellow of Clark University, to whom I am indebted for the following results:

For sample of Coil I, $\kappa = -0.986 \times 10^{-6}$

For sample of Coil II, $\kappa = -0.968 \times 10^{-6}$

¹On the Susceptibility of Diamagnetic and Weakly Magnetic Substances, *Physical Review*, 6, No. 33, April, 1898.

From these values, which agree very closely with the values found by Wills for Italian marble, it is found that for both $\mu = .999988$. Thus if all the space about the coil were filled with marble like the sample the inductance would be altered by about one part in 100,000. But only part of the air space is replaced by the marble and less than half of the magnetic lines go through the material; hence the effect of the small departure of μ from unity amounts to only a few parts in a million and is absolutely negligible.

The density of the marble is about 2.74, and its coefficient of linear expansion is about 0.000010 per degree centigrade, or about half that of brass.

3. THE GRINDING MACHINE.

The marble dealers¹ delivered the cylinders accurate to an eighth or a sixteenth of an inch. The dimensions chosen for these standards were such as to give an inductance of about 0.2 henry. For theoretical reasons the length of the coil was made equal to the radius multiplied by $\sqrt{3}$. This made the diameter about 54 cm and the length covered with wire about 46 cm, the total length of the marble cylinder being about 51 cm; the thickness of the hollow cylinder after the core was removed was about 11 cm. It is thus seen that the size and weight are not inconsiderable. The only machine in Worcester capable of grinding such a cylinder was the Poole grinder of the Rice, Barton & Fales Company.

The inductance coil consists of a single layer of wire wound on as nearly a perfect circular cylinder of marble as it is possible to produce. Every turn of wire is thus open for inspection; and there is very little electrostatic capacity. In fact, with such a coil the capacity is smaller than for any other coil of the same inductance. A circular cylinder, determined by two dimensions only, is the easiest figure to obtain perfect. The wire was of such a size as to give about 15 or 16 turns to the centimeter. With such a small pitch, it is evident that a very close approximation is made to the mathematical fiction of a cylindrical current sheet.

Cylinders of a diameter less than 45 cm can be ground also on the Norton² grinder, although I have had no experience with cylinders

¹ Torreys & Co., 69 Beverly street, Boston, Mass.

² Vid. sub.

of such large size on this machine. An accuracy of from 1/5000 to 1/10000 of an inch can be obtained on small work (say from 20 cm down) both in steel and brass. The great advantage of this machine is that the work can be revolved on centers, which is the most accurate of all methods. The disadvantage is that the precision of the work depends upon the straightness of the ways along which the grinding tool runs.

The Poole grinder is used to grind the rolls or calenders employed in paper machinery. It will grind cylinders up to 71 cm in diameter and over three meters in length, the limit in weight being about ten tons. On this grinder the cylinder being ground does not revolve on centers, but the shaft turns on bearings provided with three points of contact. It was therefore necessary to have a shaft whose bearings could be ground true on centers. This was done by the Norton Emery Wheel Company, of Worcester, Mass. The dimensions of the shaft were such that the sag of the shaft and marble when supported on its ends would produce no noticeable error in the diameter while being ground.

The deflection of a beam of circular cross section loaded at the center with a weight W is

$$\delta = \frac{Wl^3}{48EI}$$

If uniformly loaded with weight w per unit of length it is

$$\delta = \frac{5wl^4}{384EI}$$

where I is the moment of inertia of the cross section, E Young's modulus of elasticity, and l the length between supports.

Calculating by both formulæ and adding, it was found that a shaft 10 cm in diameter would deflect less than 0.025 cm, while with 12.7 cm diameter the deflection would be only about 0.01 cm. The deflection is actually much less than this, as all weights were less than those used in the calculation. The system of marble and steel was much stiffer than the simple steel shaft and the points of support nearer together than in the calculation. Besides, the bending of the shaft produces an error of the second order only, as the change in diameter of a cylinder between grinding wheels of constant distance apart, due to a sag of 0.01 cm, is very small indeed, being about one part in a million for a cylinder of 25 cm radius.

4. SHAPING THE MARBLE CYLINDERS.

Figure 1 shows the shaft with marble cylinder mounted, the total weight being in the neighborhood of three-quarters of a ton.

The Poole grinder is of special interest because it grinds the work round and makes the sides parallel, to a great extent independently of the ways on which the grinding wheels are carried. Two emery wheels are rigidly connected together, one on each side of the marble cylinder, so that as they grind at opposite ends of a diameter they practically form a grinding caliper. The cylinder, or "roll," is rotated on journals, which remain in position during the whole process. It is the system of grinding wheels which is carried back and forth on 45° angled ways.

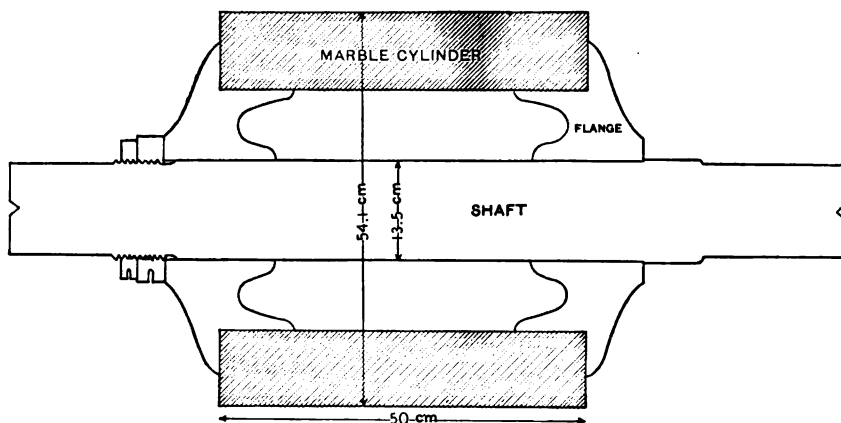


Fig. 1.—Section of Shaft and Marble Cylinder Mounted.

Consideration of this arrangement shows that if the two grinders act evenly and alike from one end of the roll to the other, which condition is indicated by the sound of the grinding, the roll must be round, and at the worst slightly conical. A cone is ground whenever the axis of the shaft of the roll is not parallel with the plane of the two ways upon which the grinding bed slides. By making this adjustment even roughly we may be sure of obtaining a practically true cylinder independently of variations in the ways, and even of the trueness of the shaft bearings.

A stream of water is continually flowing over the cylinder so that the heating effect at the grinding area is practically eliminated.

The grinders revolve at about 2,000 revolutions per minute. No pains were spared in the adjustments of the grinder to make all as nearly perfect as possible. It is easy to tell by the sound whether the emery wheels are grinding evenly or not. The amount of stock that is ground off in a single traverse is very small, not more than a thousandth of an inch. When the roll is considered nearly finished the distance between the wheels is not altered, and they are allowed to grind back and forth several times in this manner. It is not advisable to continue this process too long, as, when the wheels have no stock to grind, what is technically known as "pounding" ensues, i. e., the emery wheels rebound back and forth, cutting depressions in the marble. The grinding wheels are connected to a mass weighing about two tons, the whole suspended on knife edges, but so sensitively balanced that a slight pressure of the finger will move them.

The finished surface is smooth, bright, and almost polished, except for small scratches here and there, due to a loosening of emery particles from the wheels. No difference could be detected between any two diameters, with the cathetometer used horizontally, in the preliminary measurements. A second cylinder ground subsequently on this same machine was ground to a given diameter, as nearly as possible to within a few thousandths of an inch of the length of the end standard by which it was measured.

5. MEASURING THE CYLINDERS.

After grinding, the cylinders were transferred to the machine shop of the department of physics, Clark University, and all subsequent operations were carried on there.

A sketch of the shaft used, with the cast-iron flanged supporting disks, is shown in Fig. 1. The dimensions were such that the sag under the actual load produced no appreciable error in the grinding. It is necessary to protect the iron surface in contact with the marble with tallow, otherwise the rust will bind the two surfaces so tightly that they are separated only with great difficulty.

The measuring apparatus shown in Fig. 2 was devised so as to afford an easy method for verifying the accuracy of the cylinder. It consists of a heavy iron girder *AB* to which two brass pieces *C* and *D* are screwed. These brass pieces formed part of a geometri-

cal hinge. From these pieces hang two very stiff bars CF and DE . These bars may be taken down by pulling out two accurately ground steel rods C and D , which, passing through two triangular holes in the brass parts of the hinge, form a geometrical clamp.

The lower ends of the vertical bars were joined by a brass connecting rod EF , freely movable about axes through E and F . The system thus described forms a jointed parallelogram. The size was such as to allow about one-fourth inch clearance between the vertical bars and the cylinder when the latter was in place. Near the middle of one of the bars CF was placed a Brown & Sharpe 0.5 mm micrometer screw with a large head graduated into 100 parts.

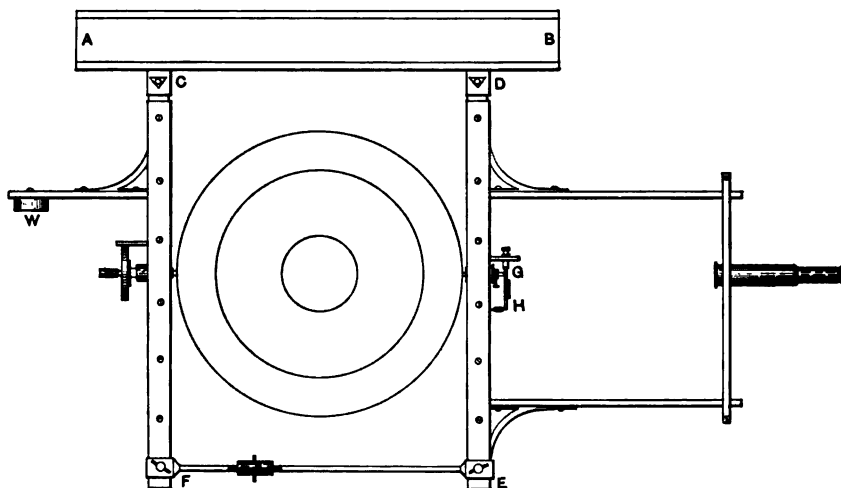


Fig. 2.—Measuring Apparatus.

Opposite, on the other bar, was placed a steel plunger, which was lightly pressed against the cylinder by means of a spring. The steel contact pieces were hardened, ground, and polished spherical.

The other end of the plunger (see Fig. 3) touches a steel lever very near to its axis of rotation. This lever is gently held against the plunger by a very light spring, and on the lever is placed a plane mirror GH . Built out from the bar is a telescope and scale attachment, by means of which any angular motion of the lever relative to the bar may be accurately read to one-tenth of a scale division.

The mean of four settings with this arrangement always agreed to within one ten-thousandth mm.

To calibrate the apparatus at any time, all that is necessary is to turn the micrometer screw through a given amount and note the readings. The multiplication was generally such that one scale division indicated about 0.0003 cm. A weight w was placed on the other bar to compensate for the weight of the telescope, etc. If now the cylinder be placed in position inside of this caliper system and turned, any variation in diameter may be observed and measured.

Bearings of hard wood were built up from the platen of a planer, and the cylinder and shaft rested on these. By means of a crank screwed to one end of the shaft the marble cylinder could be turned on its axis, and it could be moved back and forth by means of the planer. Adjustments were made once for all, so that the axis of the shaft, and therefore of the cylinder, was parallel with the ways of the planer. This could be done by trial to about 0.3 mm. A scaffolding was built up from the floor to support the measuring apparatus, so that no jarring due to motion of the cylinder could be transmitted to the frame.

It is desirable to be able to start and stop the heavy mass with small effort and without any trembling of the bed. Mutton tallow accomplishes this result, when using wood bearings, better than any other lubricant. Heavy oils ooze out, although while the cylinder is in rotation they lubricate well.

The cylinder before being set up for measurement was put in a lathe and nineteen circles 1 inch apart were drawn in pencil on the surface. Sixteen equidistant generators were also drawn, so that the whole surface was mapped out by a system of cylindrical coordinates. In measuring the diameters of any circular section, the cylinder was turned once around, thus giving two measures of each diameter; if these did not closely agree the section was remeasured. A number of surveys involving thousands of observations were made.

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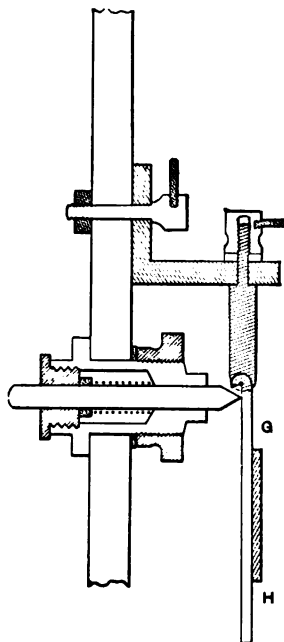


Fig. 3.—Plunger and Details of Measuring Apparatus.

The mean results are here plotted and show the remarkable perfection of the cylinders. It is probably the nearest approach to the perfect cylinder, for its size, which exists. No variation over 0.0013 cm occurs.

A particular diameter was now chosen near one end as a standard and all others referred to it. The Bureau of Standards furnished a measured steel end standard about 54.1 cm long. This bar had spherical ends, the curvature being such that the bar was a diameter of the sphere of which they form a part. This construction allows of small angular displacements of the bar without sensibly changing its length. A series of comparisons between the bar and the standard diameter on the cylinder showed an average difference of only .00011 mm. The arithmetical mean of all the diameters was now taken, giving as a result:

$$\begin{aligned}\text{Mean diameter of bare cylinder} &= \text{length of the standard bar} \\ &+ .01355 \text{ cm.}\end{aligned}$$

By a consideration of the plots of the variations in diameter along generators the mean circular section could be picked out, and by a consideration of the different diameters of this mean circular section a mean diameter could be deduced. This last estimate is entirely independent of the first value deduced by taking the arithmetical mean. The two results, however, agreed to within .00004 cm. This is a variation of less than 1 in 1,000,000. It may, therefore, safely be said that the mean diameter is correct to 1 part in 50,000, and probably to within 1 part in 100,000. Two holes were bored into the ends of the marble 180° apart and parallel to the axis; these holes were 12 in. deep, so that the temperature of the marble may be accurately determined by lowering thermometers into them. The thermometers used were all compared over a large range with a standard thermometer of Baudin, calibrated by the Bureau International at Sèvres. Holes were now bored radially through the marble, all cutting one and the same generator, and hard rubber plugs fitted in them. Measurements showed that their surfaces could be made practically continuous with the marble surface. A rough calculation will show that even were they out by a tenth of a millimeter the effect on the mean radius is infinitesimal.

6. RESULTS OF THE MEASUREMENTS.

The means of several series of measurements were plotted, and two samples of such are given below. The first (Fig. 4) shows the variations in diameter of the University coil on different circular sections, as the cylinder is turned. The sections near the ends show more variations than any other; this is due to inequalities in grinding as the wheels leave the surface at these points. The section marked

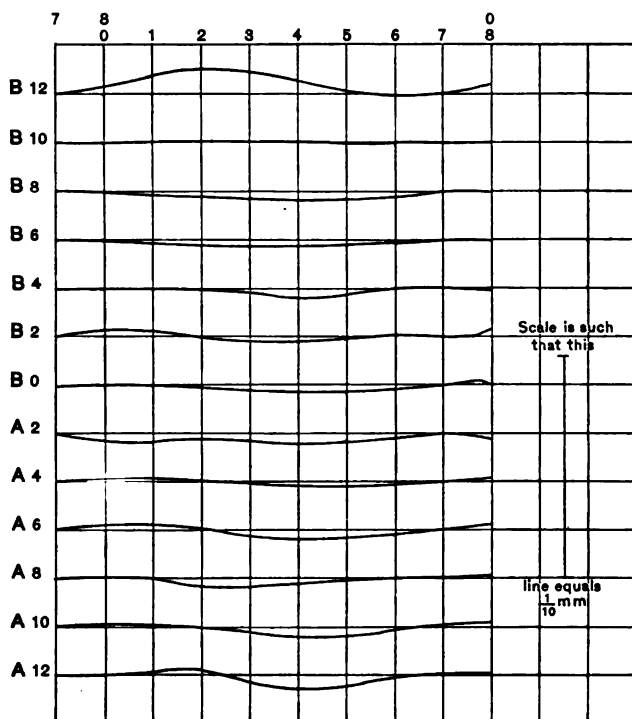


Fig. 4.—Diagrams Showing Variations in Diameters of a Number of Equidistant Circular Sections of the University Cylinder. (Abscissa 7 is an Arbitrary Zero).

B 10 happens to be so nearly circular that it is almost impossible to see any variation at all. The inequalities there are about 0.0001 mm. These curves have no relation to one another, but are all referred to an arbitrary zero, along generator 7. Fig. 5 shows the variations of diameter along different generators, as one goes from end to end, of the coil of the Bureau of Standards. The scale is such that each division is equivalent to 0.01 mm. These curves are

not independent, but show the actual relative position of every point on the cylinder, any one point being taken as an arbitrary zero.

The cylinder is seen to be on the whole very slightly conical, the amount, however, being very small, as the maximum variation is only 0.013 mm and the mean variation only about 0.005 mm.

The actual mean radius in terms of the standard is determined to a far greater accuracy than this, better, in fact, than the length of the end standard is known. The similarity of the different curves to each other shows that the cylinder is very nearly round. A plot of the variations in diameter on circular sections can easily be made by plotting the eight values, found by reading up and down on Fig. 5.

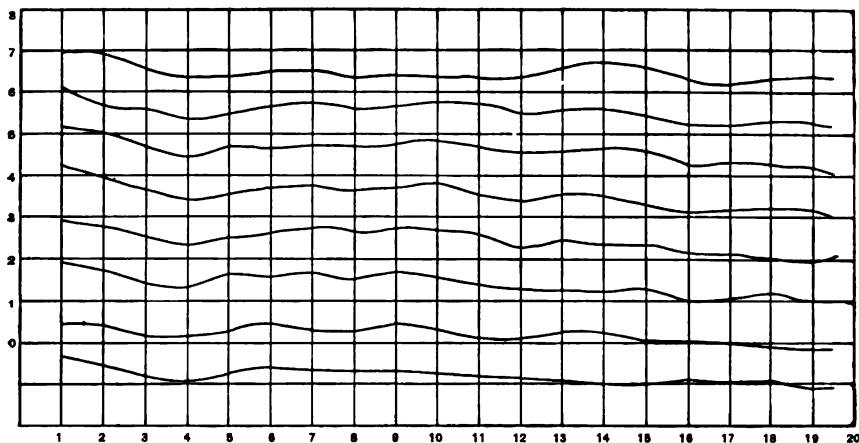


Fig. 5.—*Diagrams Showing Variations in Diameters of Bureau of Standards Cylinder. Each Space Represents 0.01 mm. The Mean Variation of All the Diameters is One-half a Space, or 0.005 mm.*

To estimate the accuracy of the wound coil numerous measurements of the variations in diameter of the completed coil were made. The variations were all under 0.01 mm. The axis of the wire has therefore less variation than this, as the above value, 0.01 mm, is due to the inequalities of four thicknesses of insulation, while the variation in the axis is only due to two.

The winding was accomplished on warm days, so that as the coil is generally at a lower temperature than that at which it was wound, the greater contraction of the copper wire over that of the marble would tighten rather than loosen it.

The length of the standard bar furnished by the Bureau of Standards at 24.21°C is $54.1140 \pm .0005$ cm. Its coefficient of expansion is 0.000011. The measurements showed that the mean diameter of the cylinder equals the length of the standard bar *minus* 0.00892 cm, the bar being at 17.50°C and the cylinder at 16.41°C . Taking the coefficient of expansion of marble to be 0.000010 (the mean of all available values for Italian marble), we find the mean diameter of the marble cylinder at 20°C as follows:

Length of standard bar at 24.21°C	=54.1140 cm
Correction for 6.71°C	= .0040
Length of standard bar at 17.50°C	=54.1100
Deduct difference between diameter of cylinder and length of bar....	= .0089
Diameter of marble at 16.41°C	=54.1011
Correction for 3.6°C , =36 parts in a million	= .0019
Mean diameter of marble cylinder at 20°C	=54.1030

7. WINDING THE COILS.

The two coils differ only in detail. In what follows the first of two values refers to the coil at Washington and the second to the coil at Clark University. The wires used are such that when wound on the cylinder there were 36.6 and 41.25 turns to an inch. These correspond to diameters of 0.0695 cm and 0.0617 cm, respectively. The thicknesses of the insulations are very small, being only 0.0030 cm and 0.0008 cm for the two coils, respectively. The wire was made expressly for this work by the General Electric Company, being drawn through a diamond die, special precautions being taken to insure a uniform thickness of insulation.

The insulation goes by the name of "Enamel," and is ideal for the present purpose. As nothing could be assumed about its insulation resistance it was necessary to measure it. Ordinary methods of insulation resistance measurements could not be used, as the insulation contained many microscopic faults which, although producing no bad effect when used dry in the ordinary manner, allowed a current to flow through the insulation into the salt water used in the test.

It was finally decided to wind two wires at a time on the University coil, so that in reality the cylinder is covered by two separate

helices of wire, every turn of one being between turns of the other, except at the ends. Measuring the resistance between the two helices the insulation resistance of the wire in place is found. This turned out to be very high, about 30 or 40 megohms between two half-mile lengths of wire in intimate contact.

Just before winding, the surface of the marble was rubbed with a solution of paraffin in turpentine to clean it and increase the surface insulation.

The wire was wound single on the Bureau of Standards coil, but the winding was subdivided into three parts, thus allowing many combinations. The insulation on the wire of this latter coil being

nearly four times thicker than on the other coil, it was thought unnecessary to wind it double, in order to test its insulating properties.

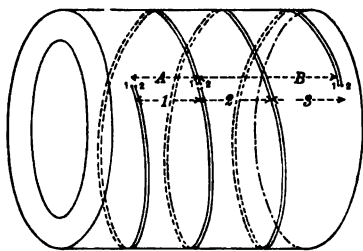


Fig. 6.—Scheme of Winding.

The scheme of winding is easily understood from the diagram, Fig. 6. To guide the wire at the start a long strip of paper was glued around the end of the marble, about seven or eight layers giving a sufficient thick-

ness. The inside edge was now trimmed with a special tool, the cylinder being turned on its own axis. In the University coil two strands were wound on at once, and the windings were pressed close together during the process. A uniform tension was mechanically maintained on the wire. The windings remained parallel to the original ones, so that after six or seven hundred turns they coincided almost exactly with the rim at the other end.

The wires go through the marble in the center of the plugs and immediately return through the same holes. This allows the coil to be subdivided and yet does not impair the uniformity of the winding; indeed it is difficult to find the spots in question where this division takes place.

In the University coil, in which the winding is double, there are in reality four separate and distinct coils alike in pairs. The terminals to these four coils are brought up to a hard rubber plug near one end of the cylinder on the inside. The wiring on the inside is

right-angled, so that the inductances of these portions may be calculated and added to the total. They are, however, very small. It is thus seen that any combination of the four coils may be made, the current flowing in either direction in any section. This gives a large number of different substandards whose values are calculable.

A standard of mutual inductance is also furnished by this coil, since if the self-inductance of the whole coil is known, as well as of the smaller coils, the following relation holds:

$$L_{1+2} = L_1 + 2 M_{12} + L_2$$

from which M_{12} may be calculated.

The coil in the possession of the Bureau of Standards is subdivided into three smaller coils all unequal. Calling these coils 1, 2, and 3, the following values have been calculated:

$$\begin{array}{lll} L_1 & L_{(1+2)} & L_{(1+2+3)} \\ L_2 & L_{(2+3)} & \\ L_3 & & \end{array}$$

From which may be derived the following:

$$\begin{array}{lll} L_{1+2} & L_{1-2} & L_{2-3} \\ L_{1-3} & L_{1+2-3} & L_{1-2+3} \\ L_{-1+2+3} M_{12} & M_{23} & \\ M_{13} & M_{(1+2)3} & M_{1(2+3)}, \text{ and others.} \end{array}$$

The ratios of the calculated values may be checked experimentally by the following process. Measure the self-inductance of two adjoining coils (1) when the current is flowing in the same direction in both, and (2) when it is flowing in opposite directions. This measurement may be made in terms of any *arbitrary* unit.

This first measurement will give—

$$L_A + 2M_{AB} + L_B = L_S = L_{A+B}$$

and the second

$$L_A - 2M_{AB} + L_B = L_O = L_{A-B}$$

where L_A and L_B are the self-inductances of the two coils and M_{AB} their mutual inductance, and where L_S and L_O denote the results of the measurements when the current is flowing in the same direction

in both and in opposite directions, respectively. We may derive from these equations the following:

$$\frac{2M_{AB}}{L_A + L_B} = \frac{L_B - L_0}{L_B + L_0},$$

the first ratio is a calculated one, while the second is an experimental one, independent of the unit used.

In a similar manner the ratio of L_A to L_B may be experimentally tested.

8. AXIAL LENGTH OF THE WINDINGS.

The axial length of the windings was measured by means of a Fuess cathetometer fitted with a micrometer microscope, the cylinder being placed on end. The cathetometer was movable about its own axis and a standard meter bar of the Société Genevoise was placed so that by a slight turning of the axis the bar was brought into the field of view. To make sure that any differences in focus might not affect the measurements the micrometer was calibrated at every reading.

As the last turns of wire were so close to the guiding rims that they could not conveniently be observed, readings were taken at noted wires near the ends and also at the windings between the separate coils into which the total coil was divided, settings being made on a fine German silver wire laid in the grooves between adjacent wires. The total number of turns being known, the total length of wire could easily be calculated. The readings were taken along sixteen different generators of the cylinder. The lengths of the subdivisions were determined with about the same absolute deviation as the total length. The wire was calipered with a Brown & Sharpe wire gauge while being wound on the cylinder. The insulation being hard enamel it was assumed not to be compressed by the tension of the wire against the marble. The diameter of the wire, as calculated from the length measurements, came out smaller than the gauge measurements, which means that the wire stretched slightly while being laid on. The computed value of the diameter of the wire is used in the calculations.

The meter bar of the Société Genevoise, used in the measurements on the University coil, had been calibrated at the Bureau of Standards for its total length, but not for fractions of its length. To verify the

calibration and obtain corrections for the fractional parts which were used to measure the length of the coil, the bar was compared with a 10 cm nickel-steel standard. This standard was furnished by the Bureau International des Poids et Mesures at Sèvres, and had such a table of corrections that any fraction of its length was completely defined. The calibration agreed to within about $5\ \mu$ with that of the Bureau of Standards, and furnished corrections as reliable as that for the fractional parts required. The mean coefficient of expansion used was:

$$\alpha_t = 10^{-8}(1844.2 + 0.66t) \text{ per degree centigrade,}$$

which was the mean of three similar meter bars calibrated and used by the Bureau International at Sèvres, these three agreeing very closely with each other.

The coil furnished to the Bureau of Standards was measured with a meter bar of the Société Genevoise, which was directly compared with a similar one whose corrections were recently furnished by the Bureau of Standards. The corrections obtained by this calibration were very small. In other words, the two meter bars were nearly identical.

Denoting the subdivisions into which the total coil was divided by the numbers 1, 2, and 3, the lengths of these sectional coils and their number of turns are as follows, correction being made for temperature and the calibration error of the bar:

Coils	Number of turns	Length at 20° C.
1	221	15.3347 cm
2	251	17.3565
3	189	13.1945
1+2+3	661	45.8857
1+2	472	32.6912
2+3	440	30.5510

The coefficients used to reduce these values were for the brass meter bar:

$$\alpha = 10^{-8}(1814.2 + 0.66t)$$

and for the coil the coefficient of expansion of *marble*, as the coil in sinking from the high temperature at which it was wound would separate the turns, and they would then move with the marble and not as if they formed a copper sheet. The mean temperature of the bar was 23°10 C, and that of the marble was 22°62 C.

The coefficient for marble was used because the wires would decrease in diameter with decreasing temperature faster than the marble would pull them together, and the amount by which they would separate would be equal to the difference in contraction of a copper over a marble cylinder divided by the number of turns minus one.

PART II.

THEORY OF THE CALCULATION OF STANDARDS OF MUTUAL AND SELF INDUCTANCE.

Six methods will be given, some of which are applicable to the calculation of mutual and others to self-inductance. These methods assume different degrees of importance, depending upon the dimensions of the coil or coils. Some are applicable to short coils only and some to long coils only, while others apply to any coil. Some are approximate, although correct to a high degree of accuracy if applied to a coil of the right shape, while two are absolute formulæ.

The conditions for the use of the different methods will be given under the description of them.

1. METHOD I.

SELF-INDUCTANCE BY SUMMATION OF A FINITE SERIES.

In Maxwell's treatise¹ will be found the following well-known expression for the mutual inductance between two coaxial parallel circles in terms of elliptic integrals of the first and second kinds to modulus k .

$$M = 4\pi\sqrt{aA} \left\{ \left(\frac{2}{k} - k \right) F - \frac{2}{k} E \right\} \quad (1)$$

$$k^2 = \frac{4aA}{(a+A)^2 + b^2}$$

¹ Elec. and Mag., Vol. II, page 339. (Third edition.)

where a and A are the radii of the circles, respectively, and b the distance between their planes.

Formula (1) above can be transformed into the following:

$$M = 8\pi \sqrt{aA} \frac{1}{\sqrt{k_1}} \left\{ F(k_1) - E(k_1) \right\} \quad (2)$$

where $k_1 = \frac{r_1 - r_2}{r_1 + r_2}$ and r_1 and r_2 are the greatest and least distances from one circle to the other, respectively, as in Fig. 7.

Taking an angle γ , such that its cosine is $\frac{r_2}{r_1}$ Lord Rayleigh has calculated the values¹ of $\frac{M}{4\pi\sqrt{aA}}$ for intervals of six minutes of angle for γ between 60° and 90° .

If the radii of the two circles are equal this angle is ϕ , the angle between the maximum and minimum lines in Fig. 7.

When $\phi = 90^\circ$, the two circles are coincident and the self-inductance of a circular filament (of zero cross-section) is infinite. When $\phi = 60^\circ$, the circles are at a distance apart equal to $\frac{2\sqrt{3}}{3}$ times their common radius. This is about 1.15 times that radius.

As for other reasons the length of the standard was nearly $\sqrt{3}$ times the radius, it follows that not every element of the present coil can be calculated by the table of Lord

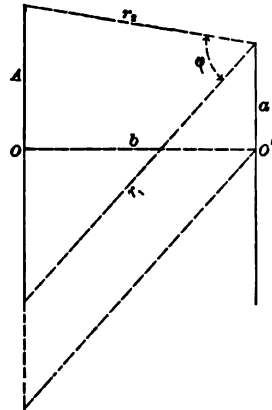


Fig. 7.

Rayleigh. A plot, Fig. 8, has been made of these values of $\frac{M}{4\pi\sqrt{aA}}$, which shows graphically how the mutual inductance of the two circles varies as we separate them from each other.

Professor Nagaoka has shown² that the mutual inductance between two parallel coaxial circles can also be expressed in terms of θ -functions, and has given the necessary formulæ. It is easy, on

¹ See table in Maxwell, page 347, and Gray, Vol. II, p. 852.

² Phil. Mag., p. 19, July, 1903.

account of the extreme rapidity of the convergence of the series he deduces, to calculate M by means of the q -series derived from the formulæ in terms of the θ -functions.

It is not possible to use Nagaoka's results to obtain a formula for the self-inductance of a coil as the necessary formulæ for the integration of θ -functions with respect to the modulus do not yet exist and would seem to be of the utmost complexity.

The self-inductance of two turns of wire is given by:

$$L_2 = 2L_1 + 2M_{12}$$

where L_1 is the self-inductance of a single turn and M_{12} is the mutual inductance between turn 1 and turn 2. In general let M_{mn} be the mutual inductance between turn m and turn n . The self-inductance of three turns is given by:

$$L_3 = 3L_1 + 2(2M_{12}) + 2M_{13}$$

Similarly for four turns

$$L_4 = 4L_1 + 2(3M_{12}) + 2(2M_{13}) + 2M_{14}$$

The law is evident, so that the expression for a coil of n turns is

$$L_n = nL_1 + 2(n-1)M_{12} + 2(n-2)M_{13} + 2(n-3)M_{14} + \dots \\ \dots + 2\{n-(n-2)\}M_{1,n-1} + 2\{n-(n-1)\}M_{1n} \quad (3)$$

The term L_1 , the self-inductance of a single turn, is obtained by finding the mutual inductance of two circular filaments at a distance apart equal to the logarithmic mean distance of the cross section of the wire from itself.

For steady currents this is 0.7788ρ , where ρ is the radius

of the cross section of the wire. For very high frequency currents it is ρ , the radius. The distance apart to be

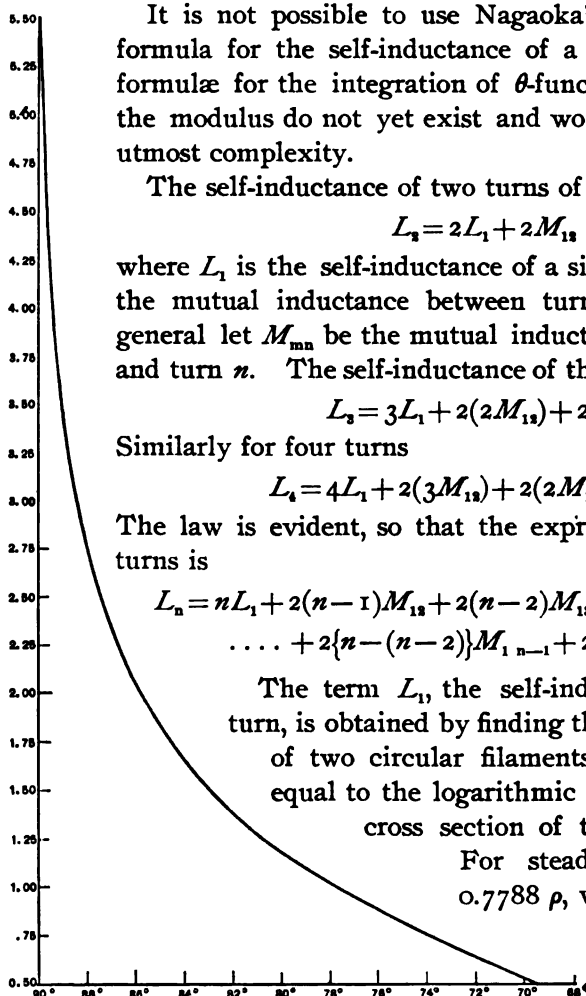


Fig. 8.—Curve. Ordinates Represent Values of $\frac{M}{4\pi\sqrt{aA}}$.

Abcissas Represent Values of the Modulus γ .

used in the calculations of the values of the M 's is the distance from axis to axis of the wires forming the two circles; furthermore this distance is the same whether the frequency is low or high.

It is thus seen that this method gives a means of computing the difference in self-inductance of a coil due to a change of frequency of the currents in it.

This method is proposed for the computation of coils with a small number of turns, say, 20 or 30. Such a computation will require as many interpolations from the table of Lord Rayleigh, as there are turns in the coil. A small portion of the table will cover the corresponding angles ϕ for such a coil. Form the first and second differences and use the interpolation formula

$$A(x) = A + x(\delta A - \frac{1-x}{2}(\delta^2 A - \frac{2-x}{3}(\delta^3 A - \frac{3-x}{4}(\delta^4 A \text{ etc.}))) \quad (4)$$

$A(x)$ is the required interpolated value, and δA , $\delta^2 A$, $\delta^3 A$ etc. are the first, second, and third differences, respectively, and x is the difference between the parameter for which the function A is desired, and the nearest one preceding it in the table.

This interpolation formula is to be found in *Légendre, Traité des Fonctions Elliptiques*. Having obtained these values, it is easy by means of a multiplying and adding machine to obtain the value of the self-inductance sought.

2. METHOD II.

SELF-INDUCTANCE BY CONVERGING INFINITE SERIES.

In Maxwell¹ will be found the following well-known series for the mutual inductance between two circles, whose radii are a and $a+y$, and whose distance apart is x .

$$M = 4\pi a \log \frac{8a}{r} \left(1 + \frac{1}{2} \frac{y}{a} + \frac{y^2 + 3x^2}{16a^2} - \frac{y^3 + 3x^2y}{32a^3} + \dots \right) \\ + 4\pi a \left(-2 - \frac{1}{2} \frac{y}{a} + \frac{3y^2 - x^2}{16a^2} - \frac{y^3 - 6x^2y}{48a^3} + \dots \right) \quad (5)$$

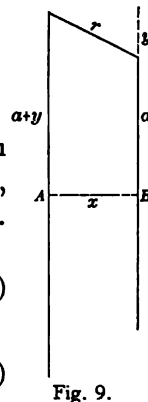


Fig. 9.

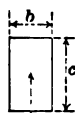
r is the least distance from one wire to the other, as shown in Fig. 9.

If the two circles be assumed to have the same radius a , the formula (5) becomes, noticing that $y=0$ and $r=x$ in this case,

$$M = 4\pi a \log \frac{8a}{x} \left(1 + \frac{1}{2} \frac{x^2}{a^2} + \dots \right) + 4\pi a \left(-2 - \frac{1}{2} \frac{x^2}{a^2} + \dots \right) \quad (6)$$

¹ Elec. and Mag., Vol. II, page 345. Third edition.

By integrating equation (5) over the cross section of a circular coil with rectangular cross section, Weinstein¹ has shown that the self-inductance of such a coil may be given by the following formula:



a is the mean radius of coil,

b is the axial breadth,

c is the radial depth.

$$L = 4\pi n^2(a\lambda + \mu)$$

where, writing x for b/c ,

$$\lambda = \log \frac{8a}{c} + \frac{1}{12} - \frac{\pi x}{3} - \frac{1}{2} \log(1+x^2) + \frac{1}{12x^2} \log(1+x^2) \\ + \frac{1}{12} x^2 \log(1+\frac{1}{x^2}) + \frac{2}{3} (x - \frac{1}{x}) \tan^{-1} x,$$



$$\mu = \frac{c^2}{96a} \left[\left(\log \frac{8a}{c} - \frac{1}{2} \log(1+x^2) \right) (1+3x^2) + 3.45x^2 + \frac{221}{60} \right.$$

$$\left. - 1.6\pi x^2 + 3.2x^2 \tan^{-1} x - \frac{1}{10} \frac{1}{x^2} \log(1+x^2) + \frac{1}{2} x^2 \log(1+\frac{1}{x^2}) \right] \quad (7)$$

This formula reduces to the special cases independently worked out by Mr. Niven and Lord Rayleigh, given below. The formula is here reprinted because, as printed in the second edition of Maxwell, it is inaccurate, although correct in the third edition.

Stefan² has very materially reduced the labor of computation by this formula by presenting it in the following form: In Table I

$$x = \frac{c}{b}$$

$$L = 4\pi n^2 \left[\left(1 + \frac{3b^2 + c^2}{96a^2} \right) \log \frac{8a}{\sqrt{b^2 + c^2}} - \gamma_1 + \frac{b^2}{16a^2} \gamma_2 \right] \quad (8)$$

Stefan³ gives the following formula, without proof, for the correction to be applied to the self-inductance of a coil to take into account the insulation of the wire:

$$\Delta L = 4\pi n^2 \left\{ \log \frac{D}{d} + 0.15494 \right\}$$

¹ Wied. Annalen, 21, pp. 329-350, 1884; Maxwell, Vol. II, Appendix, p. 350.

² Wied. Annalen, 22, pp. 107-117, 1884.

³ Wied. Annalen, 22, p. 116; 1884.

This correction is to be added to the self-inductance computed by any of the formulæ just given, where there are a number of layers of wire in the cross section of the coil.

n = total number of turns.

a = mean radius of the coil.

D = diameter of covered wire.

d = diameter of bare wire.

Maxwell, Volume II, page 329, gives the number 0.11835 instead of 0.15494.

TABLE I.

Showing Values of the Constants y_1 and y_2 of Equation (8).

x	y_1	y_2	x	y_1	y_2
0.00	0.50000	0.1250	0.50	0.79600	0.3066
.05	.54899	.1269	.55	.80815	.3437
.10	.59243	.1325	.60	.81823	.3839
.15	.63102	.1418	.65	.82648	.4274
.20	.66520	.1548	.70	.83311	.4739
.25	.69532	.1714	.75	.83831	.5234
.30	.72172	.1916	.80	.84225	.5760
.35	.74469	.2152	.85	.84509	.6317
.40	.76454	.2423	.90	.84697	.6902
.45	.78155	.2728	.95	.84801	.7518
.50	.79600	.3066	1.00	.84834	.8162

The following formulæ, equations (9) and (10), independently worked out by Lord Rayleigh¹ and Mr. Niven, may be obtained from equation (7) by giving the proper values to x . The formulæ so deduced are very good approximations when the cross section of the coil is small compared with the area inclosed by a single turn.

(a) For a coil whose axial dimension is zero (that is, a flat spiral), $b=0$, and the self-inductance is given by the formula

$$L = 4\pi n^2 a \left[\log \frac{8a}{c} - \frac{1}{2} + \frac{c^2}{96a^2} \left(\log \frac{8a}{c} + \frac{43}{12} \right) \right] \quad (9)$$

¹ Collected Papers, Vol. II, p. 15.

(b) For a coil whose radial dimension is zero (that is, a cylindrical coil of a single layer), $c=0$, and the self-inductance is given by the formula

$$L = 4\pi n^2 a \left[\log \frac{8a}{b} - \frac{1}{2} + \frac{b^2}{32a^2} \left(\log \frac{8a}{b} + \frac{1}{4} \right) \right] \quad (10)$$

Formula (10) was derived independently by the writer, not knowing in what manner it was obtained by the above-mentioned mathematicians.

(c) For a coil of circular cross section, the self-inductance is given by the following formula obtained by integration of equation (5) over a circular area,

$$L = 4\pi n^2 a \left[\log \frac{8a}{\rho} - \frac{7}{4} + \frac{\rho^2}{8a^2} \left(\log \frac{8a}{\rho} + \frac{1}{3} \right) \right] \quad (11)$$

In these three formulæ

n is the total number of turns in the coil.

a is the mean radius of coil.

b is the axial dimension.

c is the radial dimension.

ρ is the radius of circular cross-section.

These formulæ are obtained by integrating the series of equation (5) over the cross-section of the coil. In particular, formula (10) is obtained by the integration of equation (6).

In order to compute a more accurate expression for the self-inductance of a cylindrical coil of a single layer, whose axial dimension b is large compared with the radius a of the coil it was necessary to compute more terms of the series (6).

By expanding the terms of

$$\frac{M}{4\pi a} = \left\{ \left(\frac{2}{\kappa} - \kappa \right) F(\kappa) - \frac{2}{\kappa} E(\kappa) \right\}$$

in infinite series and combining them, three extra terms were obtained to the series of Maxwell, equation (6), for the case in which y is put equal to zero (the condition for equal radii).

To do this it was necessary to employ the following expansions.¹

$$\kappa^2 = \frac{4a^2}{4a^2 + x^2} = \frac{1}{1 + c^2}, \text{ if } \frac{x^2}{4a^2} = c^2$$

$$\kappa^2 + \kappa'^2 = 1 \quad \kappa' \text{ is the complementary modulus.}$$

$$\begin{aligned} F(\kappa) = & \log \frac{4}{\kappa'} + \frac{1^2}{2^2} \kappa'^2 \left(\log \frac{4}{\kappa'} - \frac{2}{1.2} \right) \\ & + \frac{1^2 3^2}{2^2 4^2} \kappa'^4 \left(\log \frac{4}{\kappa'} - \frac{2}{1.2} - \frac{2}{3.4} \right) \\ & + \frac{1^2 3^2 5^2}{2^2 4^2 6^2} \kappa'^6 \left(\log \frac{4}{\kappa'} - \frac{2}{1.2} - \frac{2}{3.4} - \frac{2}{5.6} \right) \\ & + \frac{1^2 3^2 5^2 7^2}{2^2 4^2 6^2 8^2} \kappa'^8 \left(\log \frac{4}{\kappa'} - \frac{2}{1.2} - \frac{2}{3.4} - \frac{2}{5.6} - \frac{2}{7.8} \right) \\ & + \dots \end{aligned}$$

$$\begin{aligned} E(\kappa) = & 1 + \frac{1}{2} \kappa'^2 \left(\log \frac{4}{\kappa'} - \frac{1}{1.2} \right) \\ & + \frac{1^2 3^2}{2^2 4^2} \kappa'^4 \left(\log \frac{4}{\kappa'} - \frac{2}{1.2} - \frac{1}{3.4} \right) \\ & + \frac{1^2 3^2 5^2}{2^2 4^2 6^2} \kappa'^6 \left(\log \frac{4}{\kappa'} - \frac{2}{1.2} - \frac{2}{3.4} - \frac{1}{5.6} \right) \\ & + \frac{1^2 3^2 5^2 7^2}{2^2 4^2 6^2 8^2} \kappa'^8 \left(\log \frac{4}{\kappa'} - \frac{2}{1.2} - \frac{2}{3.4} - \frac{2}{5.6} - \frac{1}{7.8} \right) \\ & + \dots \end{aligned}$$

Putting $c = \frac{x}{2a}$ the formula derived is

$$\begin{aligned} \frac{M}{4\pi a} = & \log \frac{4}{c} \left\{ 1 + \frac{3}{4} c^2 - \frac{15}{64} c^4 + \frac{35}{256} c^6 - \frac{1575}{(128)^2} c^8 + \dots \right\} \\ & + \left\{ -2 - \frac{1}{4} c^2 + \frac{31}{128} c^4 - \frac{247}{1536} c^6 + \frac{163,695}{(128)^2 \times 84} c^8 - \dots \right\} \end{aligned} \quad (12)$$

¹Gray: Absolute Meas. in Elec. and Mag., Vol. II, Part I, page 320.

The last three terms in each of the parentheses are the new terms, and form a continuation of Maxwell's formula, equation (6), for the case $y=0$. Hence, this formula gives the mutual inductance between two circles of the same radius, when placed at a distance apart equal to x .

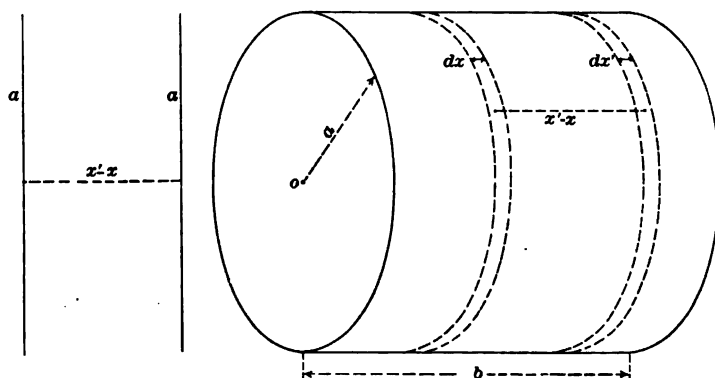


Fig. 11.

Consider a current sheet, the current in it being unity and the length of the circular cylindrical sheet being b .

Then in a length dx there will be a current $\frac{dx}{b}$ and in another element dx' a current of strength $\frac{dx'}{b}$. The mutual inductance between these two circular elements is then

$$dM = M_{x',x} \frac{dx}{b} \cdot \frac{dx'}{b}$$

where $M_{x',x}$ is the mutual inductance between two circular filaments at points whose coordinates are x' and x and hence whose distance apart is $(x' - x)$. The total self inductance of the current sheet is

$$L = \int_{-\frac{b}{2}}^{+\frac{b}{2}} \int_{-\frac{b}{2}}^{+\frac{b}{2}} M_{x',x} \frac{dx}{b} \cdot \frac{dx'}{b}$$

$$= \frac{1}{b^3} \int_{-\frac{b}{2}}^{+\frac{b}{2}} dx' \int_{-\frac{b}{2}}^{+\frac{b}{2}} M_{x',x} dx$$

Using the following formulæ of integration, which may readily be deduced by integration by parts, we obtain formula (13).

$$\int_{-\frac{b}{2}}^{+\frac{b}{2}} dx' \int_{-\frac{b}{2}}^{+\frac{b}{2}} (x' - x)^n \log (x' - x) dx = \frac{2b^{n+2}}{(n+1)(n+2)} \log b$$

$$- \frac{2(2n+3)}{(n+1)^2(n+2)^2} b^{n+2}$$

where n is integral, even and positive, and

$$\int_{-\frac{b}{2}}^{+\frac{b}{2}} dx' \int_{-\frac{b}{2}}^{+\frac{b}{2}} (x' - x)^n dx = \frac{2}{(n+1)(n+2)} b^{n+2} \text{ if } n \text{ is even,}$$

$$= 0 \text{ if } n \text{ is odd.}$$

$$L = 4\pi an^2 \left\{ \begin{array}{ll} \left(\log \frac{8a}{b} - \frac{1}{2} \right) + \frac{b^2}{32a^2} \left(\log \frac{8a}{b} + \frac{1}{4} \right) & A \\ - \frac{1}{1024} \frac{b^4}{a^2} \left(\log \frac{8a}{b} - \frac{2}{3} \right) & B \\ + \frac{10}{131072} \frac{b^6}{a^2} \left(\log \frac{8a}{b} - \frac{109}{120} \right) & C \\ - \frac{35}{4194304} \frac{b^8}{a^2} \left(\log \frac{8a}{b} - \frac{431}{420} \right) + \dots & D \end{array} \right. \quad (13)$$

The last three terms, B , C , D , are derived from the new terms of the extended equation of Maxwell. The terms marked A are the well known ones for the special case, worked out by Lord Rayleigh and Mr. Niven.

This equation is comparatively easy to work with. It applies the better the shorter the coil. For a long coil, i. e. one as long as 44

cm with a diameter of 54 cm, it gives results about three or four parts in 100,000 too small.

For such a coil the rapidity of convergence may be seen by the following set of values

$$L = 4\pi an^2 \left\{ 1.091865 + .152585 - .006352 + .000972 - .000168 \right\}$$

For short coils the convergence is very rapid. The terms for a coil of length 21 cm, diameter 54 cm, are as follows:

$$L = 4\pi an^2 \left\{ 1.849647 + .062003 - .000558 + .000022 - .0000013 \right\}$$

The formula is thus seen to be practically exact up to eight places of significant figures. This formula the writer considers the best for the calculation of the self inductance of cylindrical current sheets, as it does not require any special tables, and having once obtained the logarithms of the several numerical factors it is a matter of about forty minutes to compute the value of a coil with seven place tables correct to six places of significant figures.

For very long coils, this formula gives results a trifle too small and the computation of an extra term is desirable. It is, however, amply sufficient for the present purpose.

3. METHOD III.

SELF-INDUCTANCE IN TERMS OF ELLIPTIC INTEGRALS. EXACT FORMULA.

It is proposed to find the mutual inductance between two coaxial helices, using the Neumann formula

$$M = \iint \frac{\cos \epsilon \, ds \, ds'}{r} \quad (14)$$

where ds and ds' denote elements of arc of the respective circuits, ϵ the angle between these elements, and r their distance apart. The integral is to be taken over both circuits.

Let the equations of the two helices be respectively:

$$\begin{array}{ll} y = a \cos \theta & y' = A \cos \theta' \\ z = a \sin \theta & z' = A \sin \theta' \\ x = p \theta & x' = p' \theta' \end{array}$$

where a and A are the radii of the two helices, p and p' their respective pitches, that is the amount advanced per radian of revolution, the amount advanced in one complete turn being $2\pi p$. Neumann's formula, which in rectangular coordinates is

$$M = \iint \frac{dx dx' + dy dy' + dz dz'}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}}$$

becomes on substituting the required values

$$M = \int_{\theta_1'}^{\theta_2'} \int_{\theta_1}^{\theta_2} \frac{\{p p' + A a \cos(\theta - \theta')\} d\theta d\theta'}{\sqrt{(p\theta - p'\theta')^2 + A^2 + a^2 - 2Aa \cos(\theta - \theta')}} \quad (15)$$

This expression is unintegrable in terms of any known functions and therefore the general problem proposed is not solvable. Let us put $p = p'$, i.e., make the two helices have the same pitch, then

$$M = \iint \frac{\{p^2 + A a \cos(\theta - \theta')\} d\theta d\theta'}{\sqrt{p^2(\theta - \theta')^2 + A^2 + a^2 - 2Aa \cos(\theta - \theta')}}.$$

Change the variable, putting

$$\begin{aligned} \theta - \theta' &= \phi \\ \theta' &= \phi' \end{aligned}$$

We now have

$$M = \int_{\theta_1'}^{\theta_2'} \int_0^{\theta} \frac{(p^2 + A a \cos \phi) d\phi d\phi'}{\sqrt{p^2\phi^2 + A^2 + a^2 - 2Aa \cos \phi}}$$

We can integrate this expression with respect to ϕ' giving, if we make the number of turns an integral one, n (say),

$$M = 2\pi n \int_0^{2\pi n} \frac{(p^2 + A a \cos \phi) d\phi}{\sqrt{p^2\phi^2 + A^2 + a^2 - 2Aa \cos \phi}}$$

This integral is still unmanageable.

Putting $p' = 0$, we obtain the mutual inductance between a circle and a coaxial helix; equation (15) then becomes,

$$M = \int_0^{2\pi} \int_0^{\theta_1} \frac{A a \cos(\theta - \theta') d\theta d\theta'}{\sqrt{A^2 + a^2 - 2Aa \cos(\theta - \theta') + p^2\theta^2}}$$

This equation has been integrated by J. Viriamu Jones.

Take as new variable $\phi = \theta - \theta'$ and put $a^2 = A^2 + a'^2 - 2 Aa \cos \phi$, we have

$$M = \int_{-\theta}^{-\theta+2\pi} \int_0^{\theta_1} \frac{Aa \cos \phi \cdot d\theta \cdot d\phi}{\sqrt{p^2 \theta^2 + a^2}}$$

But we can find for every value of θ (that is, for every element of the helix) an element on the circle corresponding to a constant value of ϕ . Hence, if we integrate first with respect to θ keeping ϕ constant, and then with respect to ϕ from 0 to 2π we shall obtain the required integral. Thus,

$$\begin{aligned} M &= \int_0^{2\pi} d\phi \int_0^{\theta_1} \frac{Aa \cos \phi \cdot d\theta}{\sqrt{p^2 \theta^2 + a^2}} \\ &= \int_0^{2\pi} d\phi \frac{Aa \cos \phi}{p} \log \left\{ \frac{p\theta_1}{a} + \sqrt{1 + \frac{p^2 \theta_1^2}{a^2}} \right\} \end{aligned} \quad (16)$$

This expression is then the mutual inductance between a helix and a coaxial circle, the circle being in the plane of one end of the helix.

Prof. J. Viriamu Jones¹ has found that this integral can be reduced to the expression

$$M_{\theta_1} = \theta_1 (A + a) c k \left\{ \frac{F - E}{k^2} + \frac{c'^2}{c^2} (F - \Pi) \right\} \quad (17)$$

F , E , and Π are the complete elliptic integrals of the first, second, and third kinds, respectively, and

$$c^2 = \frac{4Aa}{(A+a)^2}, \quad k^2 = \frac{4Aa}{(A+a)^2 + b^2},$$

b = axial length of helix,

$$-c^2 = -1 + k'^2 \sin^2 \beta, \text{ and } \sin \beta = \frac{c'}{k'}.$$

To find the mutual inductance between a helix and a current sheet, both circular and cylindrical, we may proceed in the follow-

¹ Phil. Trans. Roy. Soc., vol. 182 (1891), A; Proc. Roy. Soc., vol. 63, page 192; Phil. Mag., Jan., 1889.

ing manner. Integrate the expression found for the mutual inductance between a circle and a helix, thus making the circle develop the cylindrical current sheet. If we call x the variable along the axis of the current sheet, we easily see that the mutual inductance between a circle and a coaxial cylindrical circular current sheet is given by

$$\begin{aligned}
 M &= \int_{x_1}^{x_2} dx \int_0^{2\pi} d\theta \int_0^{2\pi} d\theta' \frac{Aa \cos(\theta - \theta')}{\sqrt{A^2 + a^2 - 2Aa \cos(\theta - \theta') + x^2}} \\
 &= 2\pi \int_0^{2\pi} d\phi \int_{x_1}^{x_2} \frac{Aa \cos \phi \, dx}{\sqrt{a^2 + x^2}} \\
 &= 2\pi Aa \left[\int_0^{2\pi} \cos \phi \log(x_2 + \sqrt{a^2 + x_2^2}) d\phi \right. \\
 &\quad \left. - \int_0^{2\pi} \cos \phi \log(x_1 + \sqrt{a^2 + x_1^2}) d\phi \right] \quad (18)
 \end{aligned}$$

If the circle is in the plane of the end of the current sheet $x_1 = 0$ and the second integral of the last expression reduces to

$$\int_0^{2\pi} \cos \phi \log a \, d\phi$$

which when combined with the first integral gives

$$M = 2\pi Aa \int_0^{2\pi} \cos \phi \log\left(\frac{x_2}{a} + \sqrt{1 + \frac{x_2^2}{a^2}}\right) d\phi \quad (19)$$

Comparing this expression with equation (16) we immediately see, since $p\theta$ is the axial length of the helix, that the mutual inductance between a circle and a coaxial helix is *identically* the same as that between a circle and a coaxial current sheet of same axial length as the helix and occupying the same position. This result was obtained by Professor Jones, and appears to the writer to be a remarkable one. It leads immediately to the conclusion that the mutual inductance

between two coaxial current sheets is the same as that between a current sheet and a coaxial helix of same axial length and radius, and occupying the same position.

The writer has found an expression for this mutual inductance in terms of elliptic integrals of all three kinds.

Integrate expression (18) from $x = x_1$ to $x = x_2$

$$\begin{aligned}
 M &= \int_{x_1}^{x_2} dx \int_0^{2\pi} 2\pi Aa \cos \phi \left[\log \left(\frac{x_2' - x}{a} + \sqrt{1 + \frac{(x_2' - x)^2}{a^2}} \right) \right. \\
 &\quad \left. - \log \left(\frac{x_1' - x}{a} + \sqrt{1 + \frac{(x_1' - x)^2}{a^2}} \right) \right] d\phi \\
 &= 2\pi Aa \int_0^{2\pi} \cos \phi \cdot d\phi \left[f(x_2' - x_1) - f(x_2' - x_2) + f(x_1' - x_2) - f(x_1' - x_1) \right]
 \end{aligned}$$

where
$$f(z) \equiv z \log \left(\frac{z}{a} + \sqrt{1 + \frac{z^2}{a^2}} \right) - \sqrt{a^2 + z^2} \quad (20)$$

and therefore
$$f(z) = f(-z)$$

If the axial length of the two current sheets be denoted by $2m$ and $2l$ respectively, and the distance apart of their mean planes by x_0 , the expression becomes

$$\begin{aligned}
 M &= 2\pi Aa \int_0^{2\pi} \cos \phi \cdot d\phi \left[f(x_0 + l + m) - f(x_0 + l - m) + f(x_0 - l - m) \right. \\
 &\quad \left. - f(x_0 - l + m) \right] \quad (20a)
 \end{aligned}$$

Equation (20a), which expresses the mutual inductance between a current sheet of length $2m$ and radius A and one of length $2l$ and radius a , may be integrated in a manner similar to the simpler one actually reduced.

The problem of the mutual inductance of two coaxial cylindrical circular current sheets may then be considered to be completely solved.

If the two current sheets be made coincident in position and of the same axial length, i. e., letting $x_0 = 0$, and $2l = 2m$, and using the relation (20) above, the equation reduces to

$$M = 4\pi Aa \int_0^{2\pi} \cos \phi \cdot d\phi \left[f(2m) - f(0) \right] \quad (21)$$

As yet the two cylinders may differ in radii. Replacing $f(2m)$ and $f(0)$ by their equivalents (see eq. (20)) the following equation is obtained, and it remains to reduce it to elliptic integrals of standard form.

$$\frac{M}{4\pi Aa} = \left[\begin{array}{l} \int_0^{2\pi} \cos \phi \cdot d\phi b \log \left(\frac{b}{a} + \sqrt{1 + \frac{b^2}{a^2}} \right) \quad (P) \\ - \int_0^{2\pi} \cos \phi \cdot d\phi \sqrt{b^2 + a^2} \quad (-Q) \\ + \int_0^{2\pi} \cos \phi \cdot d\phi \sqrt{0 + a^2} \quad (R) \end{array} \right] \quad (22)$$

where b = axial length.

We may now proceed to determine the above three integrals in turn, designating them as P , Q , and R , respectively.

Integrating the first integral (P) by parts, a new expression is obtained, of which the integrated portion vanishes at the limits, giving:

$$P = b^2 Aa \int_0^{2\pi} \frac{\sin^2 \phi \, d\phi}{a^2 \sqrt{a^2 + b^2}} \quad (23)$$

Now make the following substitution

$$\psi = \frac{\pi}{2} - \frac{\phi}{2} \text{ or } \phi = \pi - 2\psi$$

Then

$$\sin \phi = \sin (\pi - 2\psi) = \sin 2\psi = 2 \sin \psi \cos \psi$$

$$\cos \phi = \cos (\pi - 2\psi) = -\cos 2\psi = 2 \sin^2 \psi - 1$$

$$d\phi = -2d\psi$$

The limits are now $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ instead of 2π and 0. If we call

$$z^2 = (A+a)^2 + b^2 \quad (24)$$

$$k^2 = \frac{4Aa}{(A+a)^2 + b^2} = \frac{4Aa}{z^2}$$

$$c^2 = \frac{4Aa}{(A+a)^2} = 1 - c'^2$$

the integral (23) may be written

$$P = \frac{16b^2Aa}{z(A+a)^2} \int_0^{\frac{\pi}{2}} \frac{\sin^2 \psi \cos^2 \psi d\psi}{(1 - c^2 \sin^2 \psi) \sqrt{1 - k^2 \sin^2 \psi}}$$

The value of this integral has been found by Jones¹ to be:

$$P = \frac{b^2(A+a)}{Aa} ck \left\{ \frac{F-E}{k^2} + \frac{c'^2}{c^2} (F-\Pi) \right\}$$

In Cayley's *Treatise on Elliptic Functions*, section 183, may be found the following relation:

$$k'^2 \sin \beta \cos \beta (F - \Pi) = -\frac{\Pi}{2} - F(k)F(k'\beta) + E(k)F(k'\beta) + F(k)E(k'\beta)$$

where

$$\sin \beta = \frac{\sqrt{1-c^2}}{k'} = \frac{c'}{k'}, \quad k'^2 = 1 - k^2$$

and

$$-c^2 = -1 + k'^2 \sin^2 \beta$$

by means of which the necessary integral Π may be calculated.

To express the second part of the integral $Q = \int_0^{2\pi} \cos \phi \sqrt{a^2 + b^2} d\phi$

$$\text{put } \psi = \frac{\pi}{2} - \frac{\phi}{2}$$

$$\text{then } \cos \psi = \sin \frac{\phi}{2}$$

$$\sin \psi = \cos \frac{\phi}{2}$$

¹ Loc. cit.

$$\cos \phi = 2 \cos^2 \frac{\phi}{2} - 1$$

$$1 + \cos \phi = 2 \cos^2 \frac{\phi}{2}$$

$$\begin{aligned} a^2 + b^2 &= A^2 + a^2 + 2aA + b^2 - 2Aa (\cos \phi + 1) \\ &= (A+a)^2 + b^2 - 4Aa \cos^2 \frac{\phi}{2} \end{aligned}$$

$$a^2 + b^2 = z^2 \left(1 - \frac{4Aa}{z^2} \sin^2 \psi \right) = z^2 (1 - k^2 \sin^2 \psi)$$

where $k^2 = \frac{4Aa}{z^2}$ and is always less than unity.

$$Q = \int_{\frac{\pi}{2}}^{-\frac{\pi}{2}} z \sqrt{1 - k^2 \sin^2 \psi} (2 \sin^2 \psi - 1) (-2 d\psi)$$

This integral being an even function we may write

$$\begin{aligned} Q &= 4z \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \psi} (2 \sin^2 \psi - 1) d\psi \\ &= 8z \int_0^{\frac{\pi}{2}} \frac{\sin^2 \psi (1 - k^2 \sin^2 \psi)}{\sqrt{1 - k^2 \sin^2 \psi}} d\psi - 4z \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \psi} d\psi \\ &= 8z \int_0^{\frac{\pi}{2}} \frac{\sin^2 \psi d\psi}{\Delta \psi} - 8zk^2 \int_0^{\frac{\pi}{2}} \frac{\sin^4 \psi d\psi}{\Delta \psi} - 4zE(k) \\ &= Q_1 - Q_2 + Q_3, \text{ say} \end{aligned} \tag{25}$$

where $\Delta \psi = \sqrt{1 - k^2 \sin^2 \psi}$

Now by Bierens de Haan's tables of definite integrals,

$$Q_1 = 8z \frac{1}{k^2} \left\{ F(k) - E(k) \right\} \tag{26}$$

and

$$Q_2 = 8z k^2 \frac{1}{3k^2} \left\{ (2 + k^2) F(k) - 2(1 + k^2) E(k) \right\} \quad (27)$$

Both of these formulæ have been verified independently by direct reduction.¹

The second part of the integral $Q = Q_1 - Q_2 - Q_3$ is therefore

$$\begin{aligned} Q &= \frac{8z}{3k^2} \left\{ 3F - 3E - \frac{3}{2} k^2 E - (2 + k^2) F + 2(1 + k^2) E \right\} \\ &= \frac{4z}{3k^2} \left\{ 2(1 - k^2) F(k) + (k^2 - 2) E(k) \right\} \end{aligned} \quad (28)$$

R is expressed in the same way that Q is, excepting that b is put equal to zero, which gives $z = A + a$ and $k_0^2 = \frac{4Aa}{(A+a)^2} = c^2$. Inserting these values, we obtain

$$R = \frac{2(A+a)}{3Aa} \left\{ (A-a)^2 F(k_0) - (A^2 + a^2) E(k_0) \right\} \quad (29)$$

The problem then by this method is completely solved, as we have

$$\frac{M}{4\pi Aa} = P - Q + R$$

It was at first thought that instead of putting $A = a$ for the coefficient of mutual inductance of two coincident current sheets, which would give the self-inductance of a single such current sheet, it might be found more accurate to compute the mutual inductance of two such current sheets of radii $a + r/2$ and $a - r/2$, where r is the logarithmic mean distance of the cross section of the wire, a circle, from itself. This is given² by $r = 0.7788\rho$ where ρ is the radius of the section of the wire. More accurately it is $r = \rho e^{-\pi/4}$.

By direct calculation it is found that this refinement is unnecessary, the result being the same to seven significant figures whether a be put equal to A , or whether the above suggestion be followed.

¹ Jordan, *Traité d'Analyse*, II, 22.

² Maxwell, *Elec. and Mag.*, Vol. II, page 324. Third edition.

There is, however, a direct and important gain in simplicity by putting $a = A$.

In this case $c = 1$ and $c' = 0$ in equation (24).

The equation for P then simplifies into

$$P = \frac{4b^3}{z} \frac{F-E}{k^3}$$

$$Q = \frac{4z}{3k^3} \left\{ 2(1-k^2)F - (2-k^2)E \right\} \quad (30)$$

and

$$R = -\frac{8}{3}a$$

$$k^2 = \frac{4a^2}{4a^2 + b^2} \text{ and } z = \sqrt{4a^2 + b^2}$$

Hence

$$L' = 4\pi a^2 \{P - Q + R\}$$

This is the self-inductance on the assumption that the current per unit length of current sheet is unity. If there are n_1 turns of wire per unit of length in the coil, and unit current flows in each turn, the corresponding current in the current sheet would be n_1 units instead of one, and as this current enters to the second power the self-inductance of a cylindrical coil is

$$L = 4\pi a^2 n_1^2 \{P - Q + R\}$$

We may replace n_1 by $\frac{n}{b}$ and the equation becomes finally

$$L = 4\pi a^2 \frac{n^2}{b^2} \left\{ \frac{4b^3}{\sqrt{4a^2 + b^2}} \left(\frac{F-E}{k^3} \right) - \frac{4\sqrt{4a^2 + b^2}}{3k^3} \left(2(1-k^2)F - (2-k^2)E \right) - \frac{8}{3}a \right\} \quad (31)$$

This formula involves finding the *complete* elliptic integrals of the first and second kinds only, a fairly easy matter, using Legendre's tables.

Calling $2a=d$ this equation may be reduced to the following, suggested by the formula of Method IV. Hence, for the case considered the above equation (31) and the corresponding one of method VI are identical and equal to

$$L = \frac{4}{3} \pi \frac{n^2}{b^3} \left\{ p(d^2 - b^2)E + pb^2F - d^2 \right\} \quad (32)$$

where

d = diameter of cylinder = $2a$

b = length of cylinder

$$p^2 = d^2 + b^2$$

$$k^2 = \frac{d^2}{d^2 + b^2} = \frac{d^2}{p^2}$$

k is the modulus of the elliptic integrals E and F .

By computing, however, with both formulæ (31) and (32) a valuable check on the work is obtained. This was done by the writer before the equivalence of the two formulæ was discovered.

This equivalence also, of course, gives a check upon the theoretical accuracy of both the formulæ of this method and that of method IV.

4. METHOD IV.

MUTUAL AND SELF INDUCTANCE BY MEANS OF ELLIPTIC INTEGRALS, DUE TO KIRCHHOFF.

The following formula is an unpublished formula of Kirchhoff's for the mutual inductance between two coaxial cylindrical current sheets, symmetrically placed:

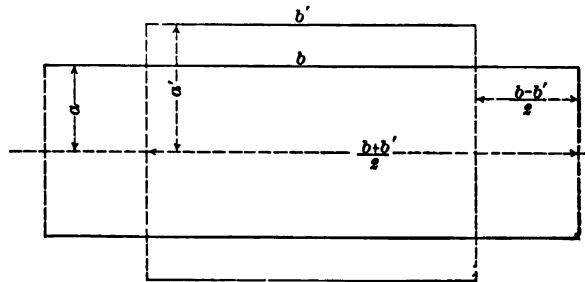


Fig. 12.

Let a and a' be the radii of the internal and external solenoids, respectively.

b and b' the corresponding lengths.

n and n' the total number of turns in each.

Then the mutual inductance between the two solenoids is given by the following expression:

$$M = 4\pi^2 \frac{n}{b} \frac{n'}{b'} a^2 b' + (V_1 - V) \quad (33)$$

where

$$\begin{aligned} V = 4\pi^2 \frac{n}{b} \frac{n'}{b'} \pi \left\{ (a' - a) c^2 \left[(F(q) - E(q)) F - F(q) E \right] \right. \\ + \frac{\sqrt{c^2 + (a' + a)^2}}{3} \left[2(a'^2 + a^2) - c^2 \right] E \\ + \frac{c^4 + 2c^2(a'^2 + a^2) - 2(a'^2 - a^2)^2}{3\sqrt{c^2 + (a' + a)^2}} F \\ \left. + ca^2 \pi \right\} \quad (34) \end{aligned}$$

and where

$$c = \frac{b - b'}{2} \text{ in the expression for } V$$

and

$$c = \frac{b + b'}{2} = c' \text{ in the expression for } V_1$$

F and E are the *complete* elliptic integrals of the first and second kinds respectively, to modulus k , where

$$k^2 = \frac{4aa'}{c^2 + (a' + a)^2}$$

$F(q)$ and $E(q)$ are the *incomplete* elliptic integrals of the first and second kinds, respectively, to modulus k_1 given by

$$k_1^2 = \frac{c^2 + (a' - a)^2}{c^2 + (a' + a)^2} \text{ their amplitude being given by the following}$$

equations:

$$\sin am\ q = \frac{c}{\sqrt{c^2 + (a' - a)^2}}; \quad \cos am\ q = \frac{a' - a}{\sqrt{c^2 + (a' - a)^2}}$$

$$\Delta am\ q = \frac{a' + a}{\sqrt{c^2 + (a' - a)^2}}$$

These integrals may all be found in the tables of Legendre, *Traité des Fonctions Elliptiques*, Tome II.

This formula, which is an exact one, is very valuable, as it expresses the mutual inductance between two symmetrically placed solenoids in terms of elliptic integrals of the first and second kinds only.

It is shown below to be equivalent to the equation of method III for the case in which $b = b'$.

It is to be noticed that it is more general than the formula of method III.

To find the self-inductance of a solenoid, we apply this formula to the case of two identical solenoids identically placed, and compute their mutual inductance. The formula simplifies very materially under these circumstances as

$$\frac{b - b'}{2} = 0 = c \text{ and } \frac{b + b'}{2} = c' = b.$$

The term containing the incomplete integrals drops out entirely since it contains the factor $(a' - a)$ which is zero. Putting $2a = d$ we have

$$V_1 = 4 \frac{n^2}{b^3} \pi \left\{ \frac{\sqrt{d^2 + b^2}}{3} (d^2 - b^2) E + \frac{b^3}{3} \sqrt{d^2 + b^2} F - \frac{bd^2\pi}{4} \right\}$$

$$V = 4 \frac{n^2}{b^3} \pi \frac{d^3}{3} E = 4 \frac{n^2}{b^3} \pi \frac{d^3}{3} \quad (35)$$

The term involving F drops out in V , since the fraction multiplying it contains in the numerator either c , which is zero, or $a' - a$ which is also zero.

Notice also that the last term of V , equation (35), just cancels the first term of equation (33), leaving the following equation for the self-inductance:

$$L = \frac{4}{3} \pi \frac{n^2}{b^3} \left\{ p(d^2 - b^2)E + p\delta^2 F - d^3 \right\} \quad (36)$$

where the letters have the meanings already given.

This formula, as has been said before, is identical with the one deduced in method III, and therefore they afford a check on each other, not only numerically, but theoretically, as showing the correctness of the improved formula of Kirchhoff.

The formula in its complete form was communicated to me by Prof. Antonio Ròiti, of Florence, to whom it was communicated by Professor Kirchhoff in writing.

Professor Ròiti¹ employed it in his calculation of the mutual inductance of the coils used in his determination of the ohm in absolute value.

5. METHOD V.

MUTUAL INDUCTANCE BY MEANS OF A CONVERGING INFINITE SERIES, DEDUCED BY INTEGRATION OF A SERIES IN SPHERICAL HARMONICS, DUE TO MAXWELL.

The self-inductance of an infinitely long solenoid per unit of length, where n_1 is the number of turns per unit length, is

$$L = 4\pi n_1^2 S$$

S being the area of the cross section of the solenoid.

The mutual inductance between two coaxial circular cylindrical solenoids of infinite length per unit of length is

$$M = 4\pi n_1 n_1' S$$

where S is the area of the smaller section; n_1 and n_1' are the number of turns per unit length for the two coils, respectively.

For a coil of finite length we must take into account the demagnetizing effect of the ends. These ends act, as is well known, like the ends of a magnet uniformly polarized axially. Since the ends cause a demagnetizing effect, or in other words a decrease in the induction because the field through one coil, due to unit current in the other, is decreased, it follows that the expression

$$L = 4\pi n_1^2 S b$$

¹ Il Nuovo Cimento, vol. 12, page 60, 1882, and vol. 15, page 97, 1884.

is an upper limit to the value for the self-inductance of a solenoid of length b , area S and n_1 turns per unit of length.

The longer the coil, in comparison with the area of its cross section, the more nearly does this give the true value.

It is interesting to note that, since $S = \pi a^2$, the foregoing approximate formula for the self-inductance of a long coil may be written

$$L = 2\pi a n_1 \times 2\pi a n_1 b \quad (37)$$

or in words the easy rule:

The self-inductance of a long solenoid is approximately given by multiplying the length of wire in unit length of the coil, by the total length of wire in the coil.

The following demonstration is taken from Maxwell:¹ In Thomson and Tait's treatise on Natural Philosophy, article 546, Example II, is to be found the following expression for the potential¹ of a plane circular magnetic shell, at a point $P(x, y, z)$, in terms of spherical harmonics.

The P 's are those zonal harmonics corresponding to the angle θ shown in the figure.

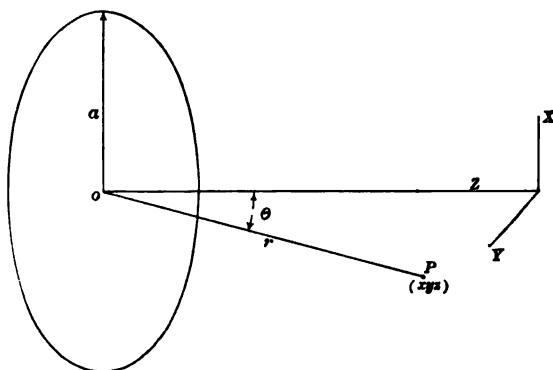


Fig. 13.

$$V = 2\pi \left\{ -r P_1 + a + \frac{1}{2} \frac{r^3}{a} P_3 - \frac{1}{2} \frac{1}{4} \frac{r^5}{a^3} P_5 + \frac{1}{2} \frac{1}{4} \frac{3}{6} \frac{r^7}{a^5} + \dots \right\} \quad r < a$$

$$= 2\pi \left\{ \frac{1}{2} \frac{a^3}{r} - \frac{1}{2} \frac{1}{4} \frac{a^5}{r^3} P_3 + \frac{1}{2} \frac{1}{4} \frac{3}{6} \frac{a^7}{r^5} P_5 - \dots \right\} \quad r > a$$

¹Electricity and Magnetism, II, p. 311.

If we differentiate the second series with respect to r , we shall obtain the radial component of the force. Now multiply by $2\pi r^2 d\mu$ and integrate from $\mu=1$ to $\mu=\frac{z}{\sqrt{z^2+A^2}}$ where A is the radius of the larger of the two coils. This gives the coefficient of inductance of a single winding of the outer solenoid due to the circular disk which forms the end of the inner solenoid at a distance z from the positive end.

To find it for all the windings integrate from $z=0$ to $z=b$, the length of the solenoids. Multiplying this by $n_1 n_1'$ the effect of one end of the inner solenoid is obtained in diminishing the coefficient of mutual inductance as given for the case of infinite length. The result is:

$$M = 4\pi^2 n_1 n_1' a^2 (b - 2Aa) \text{ where}$$

$$a = \frac{1}{2} \frac{A+b-r}{A} - \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{1}{3} \frac{a^2}{A^2} \left(1 - \frac{A^2}{r^2}\right)$$

$$+ \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \frac{1}{4} \cdot \frac{1}{5} \frac{a^4}{A^4} \left(-\frac{1}{2} - 2 \frac{A^2}{r^2} + \frac{5}{2} \frac{A^4}{r^4}\right) + \dots \quad (38)$$

where $r^2 = b^2 + A^2$.

It appears then that when the solenoids are not infinitely long, the length b must be diminished by Aa at each end.

When the solenoid is quite long compared with its radius

$$a = \frac{1}{2} - \frac{1}{16} \frac{a^2}{A^2} - \frac{1}{128} \frac{a^4}{A^4} + \dots$$

In computing self-inductance $a=A$ and $a=0.4297=0.43$ approx. Hence the self-inductance of a solenoid may be computed by the simple formula

$$L = 2\pi a n \times 2\pi a n b' \quad (39)$$

Where b' , the value used in the calculation, is the length b diminished by 0.86 times the radius.

Applying formula (38) to a coil of the following dimensions we obtain these results:

$$\begin{aligned} \text{radius, } a &= 27.09 \text{ cm} \\ \text{length, } b &= 44.12 \text{ cm} \\ n &= 716 \text{ turns.} \end{aligned}$$

The inductance found is 0.2163 henry, whereas the true inductance is 0.2145 henry. The correction to the length in this calculation is 15.99 cms.

The simpler expression (39) gives results to only 30 per cent for this case, but for long coils of length over ten times their diameter it is accurate enough for rough experimental purposes.

It is thus seen that the equation (38) gives results on such a coil as the one described, for example, to about 1 part in 100.

The formula of this method does not converge very rapidly for coils as short as those to which it had to be applied, so that the labor of extending it, while not very great, was thought unwarranted.

6. METHOD VI.

MUTUAL INDUCTANCE. RÒITI'S FORMULA.

Professor Ròiti has kindly communicated to me the following unpublished expansion of Maxwell's formula of method V, which he used in calculating the mutual inductance of two coils used in his determination of the ohm. It has not been found convenient as yet to verify this formula, but it looks very much simpler for numerical calculation than the one just given in method V.

$$M = 4\pi^2 a^2 \frac{n}{b} \frac{n'}{b_1} \left\{ \rho_1^2 - \rho_1 + \frac{a^2 A^2}{8} \left(\frac{1}{\rho_1^3} - \frac{1}{\rho_2^3} \right) - \frac{a^4 A^4}{16} \left(\frac{1}{\rho_1^5} - \frac{1}{\rho_2^5} \right) + \frac{5a^4 A^4}{64} \left(\frac{1}{\rho_1^7} - \frac{1}{\rho_2^7} \right) - \dots \right\} \quad (40)$$

where $\rho_1^2 = \left(\frac{b-b_1}{2} \right)^2 + A^2, \quad \rho_2^2 = \left(\frac{b+b_1}{2} \right)^2 + A^2$

and where

a and A are the internal and external radii, respectively,
 b and b_1 are the respective lengths,
 n and n' are the respective total numbers of turns.

The longer the coils in comparison with their radii the faster does this series converge.

7. DATA FOR CALCULATION.

ARBITRARY CORRECTION TO THE LENGTHS.

The coil lengths as given above (p. 103) were not used in the calculations. The length of any coil was taken to extend from the center of gravity of the last semicircle of wire on one end to the corresponding point in the last semicircle at the other end. The reasons for this are as follows: To take the length of coil from edge to edge of wire is evidently taking a superior limit to the length, while to take the length to extend from the axis of the last wire on one end to the axis of the last wire on the other is evidently using an inferior limit to the length that should be employed. As for low frequencies the current is almost uniformly distributed throughout the cross section of the conductor, and it is the mutual effect of the



Fig. 14.

current filaments which produces the self-inductance, the *assumption* made above seemed to the writer to be nearer the truth than either of the above extreme cases.

This correction gave the value 0.0365 cm to be subtracted from each coil length.

It occurred to the writer that this arbitrary but reasonable assumption could be tested by a reduction to an extreme case. Suppose the coil reduced to *one* turn, at what distance apart may we consider the current filaments to be concentrated so that the mutual inductance between them will give the actual self-inductance of the one turn? The theory of the logarithmic mean distance says that this distance is given by $0.7788 \times$ radius of the wire, which for this case is 0.0248 cm; but according to the assumption above made this distance would be $2 \times$ radius of the bare wire— $0.0365 \text{ cm} = 0.0269 \text{ cm}$, which is in pretty good agreement with the value 0.0248 derived above.

Or the following argument might be tried. What is the average length of all the infinitely thin current sheets into which the area of Fig. 14, considered as a current sheet with round ends, might be divided? Calculation shows that the average length would be obtained by subtracting 0.0138 cm from the total length.

How well does this apply in the limiting case of one turn? The average length of current sheets into which a circle the size of the wire may be divided is 0.0496 cm.

What is the length of the current sheet equivalent to the circle? The theory of the logarithmic mean value says that the circle is equivalent to two filaments at a distance apart of 0.0248 cm; and it also asserts that the current sheet equivalent to this is given by the equation

$$.0248 = b \times 0.2231$$

whence $b = \frac{.0248}{.2231} = 0.111$ cm and the agreement is not nearly so good.

As the whole thing is arbitrary the former assumption has been chosen as preferable to the latter and the lengths corrected accordingly.

Taking account of the thickness of the insulation at both ends, the correction to be subtracted from each coil length is

By the above theory.....	0.0365 cm
Two thicknesses insulation	.0060
Total correction.....	.0425

The difference in length in the current sheets calculated on these two assumptions is 0.02 cm, or about 1 part in 2,000 of the total length. This means about 1 part in 4,000 difference in the self-induction.

Should a better value be suggested, the calculated values may be corrected.

The average deviation of the mean of the length measurements considered as measurements of a single invariable length, was for the longest coil only about 0.005 cm, and for the shortest coil 0.003 cm, which is a degree of precision better than 1 part in 9,000 and 1 in 4,000 for the two coils, respectively. In other words, the end windings actually deviated from the mean length by only these amounts on the average.

The value of the mean length is of course much more accurately determined than this, as the average of the different measurements represents the true mean length of the coil better than the average deviations of them from the mean length as measured.

The data used in the calculations is contained in the following table. The lengths of the sections are obtained from those on page 103 by subtracting 0.0425 cm, and the diameter is obtained from the value given on page 99 by adding 0.0694, the thickness of the covered wire.

TABLE II.

Showing the Dimensions of the Several Sections of the Bureau of Standards Inductance Standard.

Coil	Number of turns	Length in cm	Diameter in cm
1	221	15.2922	54.1724
2	251	17.3140	54.1724
3	189	13.1520	54.1724
1+2+3	661	45.8432	54.1724
1+2	472	32.6487	54.1724
2+3	440	30.5085	54.1724

The ohmic resistances of the three coils are approximately as follows:

Coil 1.....	21. 10 ohms
Coil 2.....	23. 85
Coil 3.....	17. 76
Total	62. 71

8. CLARK UNIVERSITY COIL.

The same general considerations apply to the measurements, calculations, and data of the coil retained by Clark University.

The diameter compared with the standard rod was 0.00125 cm larger than the *mean* diameter. The comparison of the rod with the cylinder gave the following result.

Diam. $(\bar{B}.8, 6.5) = \text{rod} + .0148 \text{ cm}$, with an average deviation of the mean of .00011 cm, and

$$\begin{aligned}\text{mean diam} &= \text{rod} + .0148 \text{ cm} - .0013 \text{ cm} \\ &= \text{rod} + .0135 \text{ cm}.\end{aligned}$$

The cylinder and rod were both at the same temperature, 19°6 C, and as the expansion of marble and steel is practically the same,

these numbers require no correction to reduce to 20°, so that since the length of rod is 54.1115 at 20° C, the mean diameter of the bare cylinder at 20° is

$$54.1115 \text{ cm} + 0.0135 \text{ cm} = 54.1250 \text{ cm}$$

The diameter of the covered wire was .0617 cm

Of the bare wire, .0602 cm

Twice insulation, .0015 cm

The length measurements gave the following results reduced to 20° C (marble was at 25°36 C, scale at 26°10 C).

Total coil length at 20° C = 44.1548 cm, 716 turns.

Longer half of coil at 20° C = 23.4397 cm, 380 turns.

Shorter half of coil at 20° C = 20.7151 cm, 336 turns.

In the first of these determinations the average deviation of the separate measurements from the mean was 8 parts in 430,000; the remaining two had deviations of about 8 parts in 200,000.

Correcting these lengths as explained above, 0.0369 cm is to be subtracted from each coil length.

The data as used for calculation are here tabulated:

TABLE III.

Showing the Dimensions of the Sections of the Clark Standard of Inductance.

Coil	Number of turns	Length coils from edge to edge	Corrected for calculation	Diameter
A	380	23.4397 cm	23.4028 cm	54.1867 cm
B	336	20.7151	20.6783	54.1867
A+B	716	44.1548	44.1179	54.1867

Using these data the following values of self-inductance were obtained for the three coils, in henrys:

TABLE IV.

Showing the Values of Inductances of the Separate Sections, Computed by Three Different Formulae.

Coil	Method I	Method II	Method III
A_{1+2}	0.0728935 henry	0.0728924 henry	0.0728932 henry
B_{1+2}	0.0876479	0.0876483	0.0876488
$(A+B)_{1+2}$	0.216234 ¹	0.216240	0.216246

¹ This value is about 4 parts in 100,000 too small, as is expected in the use of the logarithmic formula.

The ohmic resistances of the windings are given below:

Coil A_1	21.92 ohms
Coil A_2	21.91
Coil B_1	19.83
Coil B_2	19.33
Total $(A+B)_{1+2}$	82.99

The calculations were made by seven-place logarithms. Three formulæ were at first employed, which for convenience are given below with references to places where they occur in the text. However, as two of these formulæ (II and III) were found to be identical, the later calculations were made with but the first two.

In the following formulæ,

d = diameter of the winding,
 a = radius of the winding,
 b = length of the winding,
 n = whole number of turns of wire,

$$N = \log_e \frac{8a}{b}$$

$$\text{I. } L = 4\pi a n^2 \left\{ \left(N - \frac{1}{2} \right) + \frac{b^2}{32a^2} \left(N + \frac{1}{4} \right) - \frac{1}{1024} \frac{b^4}{a^4} \left(N - \frac{2}{3} \right) \right. \\ \left. + \frac{10}{131072} \frac{b^6}{a^6} \left(N - \frac{109}{120} \right) - \frac{35}{4194304} \frac{b^8}{a^8} \left(N - \frac{431}{420} \right) \right\} \quad \text{See page 113.}$$

$$\text{II. } L = \frac{4}{3} \pi \frac{n^2}{b^2} \left\{ p(a^2 - b^2)E + pb^2F - a^2 \right\} \quad \text{See page 124.}$$

In formula II, $p^2 = a^2 + b^2$ and the complete elliptic integrals E and F are to the modulus $k^2 = \frac{a^2}{a^2 + b^2} = \frac{a^2}{p^2}$

$$\text{III. } L = \frac{4}{3} \pi a^2 \frac{n^2}{b^2} \left\{ \frac{3b^2}{pk^2} (F - E) - \frac{p}{k^2} \left[2(1 - k^2)F - (2 - k^2)E \right] - a \right\}$$

See page 123.

A sample calculation by means of I and II is here given. The labor is much diminished by carefully planning these calculations and using the correct number of significant figures. The values of the logarithms of all the constants of the formulæ are also given.

USING FORMULA (II).

Using the tables of Legendre, Vol. II, page 235, and interpolation formula of page 107.

$d^2 = 2936.199$ $b^2 = 547.688$	$x = .40904$ $\frac{2}{1-x} = 0.29548$				
$d^2 + b^2 = 4483.887$	For $66^\circ.6 \log E = 0.060326333$ — 142666	1st diff = —347338 + 20	2d diff = —68; — $68 \times 0.3 = -20$	$\log 347318 = 5.5407273$ $\log .40904 = 1.6117616$	
$d^2 - b^2 = 2388.511$	For $\theta \log E = 0.060184267$	—347318		$\log (\text{prod}) = 5.1524889$ 718	
$\log d^2 = 3.4677854$ $\log d^2 + b^2 = 3.5426641$				3d diff = +16 171	
$\log k^2 = 1.9257213$ $\log k = 1.9628606 +$ $= \log \sin \theta$	For $66^\circ.6 \log F = 0.373987685$ + 279167	1st diff = +683236 —737	2d diff = +2503 —8		
$\theta = 66^\circ 38' 20 + \frac{660''}{91}$ $= 66^\circ 38' 45.421$ $= 66^\circ .6409935$	For $\theta \log F = 0.374266852$	682499	2495		
			$2495 \times 0.295 = 737$	$\log 682499 = 5.8341021$ $\log .40904 = 1.6117616$	
$\log E = 0.0601943$ $\log (d^2 - b^2) = 3.3781272$ $\log p = 1.7710320$	$\log F = 0.3742669$ $\log b^2 = 2.7385342$ $\log p = 1.7710321$			$\log (\text{prod}) = 5.4458637$ 532	105
$\log (\text{prod } 1) = 5.2093435$	$\log (\text{prod } 2) = 4.8838332$				
$\text{prod } 1 = 161936.03$ $\text{prod } 2 = 76530.27$	$\log b^2 = 2.7385342$ $\log L_A = 7.9427420$				
$\text{Sum} = 238466.30$ $d^2 = 159102.87$	$L_A = 0.0876480 \text{ henry.}$				
$\text{Diff} = 79363.43 = G$					

9. TEMPERATURE COEFFICIENT OF THE INDUCTANCE STANDARDS.

In very accurate work variation of temperature may have a decided influence. The only way it can affect the self-inductance of the standard coil is by a change in the length and the mean radius.

These vary, for reasons stated above (page 104), with the same temperature coefficient as that of marble, namely, about 0.00001 per degree centigrade.

It is evident that these changes in dimensions may be allowed for and the temperature coefficient of inductance computed for any given coil.

To find the magnitude of this coefficient the following calculations were undertaken on coil A, of the Clark standards.

The self-inductance was calculated, making a change of 0.1 per cent in the length but keeping the radius constant, and a coefficient

$\frac{\partial L}{\partial b}$ was thus determined. Then by making an equal change in the radius and keeping the length constant $\frac{\partial L}{\partial a}$ was determined.

Then evidently,

$$dL = \frac{\partial L}{\partial a} da + \frac{\partial L}{\partial b} db.$$

for simultaneous changes in a and b as long as they are small.

The results were that

.1 per cent increase in a produced .152 per cent increase in L

.1 per cent increase in b produced .052 per cent decrease in L

This means that

$$\frac{\partial L}{\partial a} = .00491 \text{ henrys per cm}$$

$$\frac{\partial L}{\partial b} = -.00194 \text{ henrys per cm}$$

This shows that the length measurements on the radii should be about 2.5 times as precise as those on the length to give the same relative error in the computations.

A change of 1 in 1,000 in both a and b would produce a joint error in L of about 1 in 1,000. So that the inductance temperature coefficient of this particular coil is about $+.00001$ per 1°C , as the linear temperature coefficient is about $+.00001$ per 1°C . The temperature coefficients of the other coils are approximately of this same order. A reference to the table below will show just how the coefficient $\frac{\partial L}{\partial a}$ varies for coils of different lengths. The coefficients $\frac{\partial L}{\partial b}$ have not as yet been calculated.

Values of $\frac{\frac{\partial L}{\partial a}}{\frac{L}{a}}$. 1% change in a produces $x\%$ change in L .

Coil	$x\%$	b
1	1.436	15.29 cm
2	1.465	17.31
3	1.411	13.15
1+2+3	1.646	45.84
1+2	1.582	32.65
2+3	1.554	30.51

10. ON THE DISTRIBUTED CAPACITY OF A COIL.

Distributed capacity must not be confounded with the ordinary electrostatic capacity of an insulated conductor. A coil of wire considered as an insulated conductor, has, of course, a definite electrostatic capacity, which is numerically measured by the quantity of electricity necessary to raise the coil to unit potential.

To explain the idea of distributed capacity, consider a coil of a single layer in which an electrical charge is oscillating. The period of this oscillation has a value given by the formula

$$T = 2\pi \sqrt{L_m C_m}$$

where T is the period of a complete oscillation to and fro in seconds; L_m the self-inductance of the coil for high frequency currents in henrys and C_m the distributed capacity of the coil in farads. If T and L_m are known, then the value of C_m may be computed by the formula. This capacity, C_m , arises in the following manner:

On account of the rapidity of the oscillations, the distribution of the moving charge is not uniform; this causes different portions of the

same coil to be at different potentials for the time being, so that two adjacent turns being at different potentials, a charge will accumulate on them, for the instant, just as if they were insulated conductors in the same relative position and charged to the same difference of potential. Strictly speaking, every portion of the wire would be at a different potential from every other portion, and to be exact one should speak of adjacent portions of adjacent turns. The distribution of the potential upon the wires is connected with the period, so that it follows that the distributed capacity is some function of the period. Such conditions give rise to a true electrostatic capacity which helps determine the frequency of the oscillations. The calculation of the value of this capacity is evidently a very complex matter. It would be much larger for coils of many layers than for a coil of a single layer with the same number of turns.

Paul Drude,¹ in a most instructive and valuable paper, makes an approximate theoretical calculation, thoroughly checked by experiment, upon the value of the wave length λ for the fundamental oscillation in a coil of a single layer. By means of his results, the fundamental periods for the two coils here considered may be calculated. The value of this capacity for single-layer coils is very small. Lodge and Glazebrook² estimated the value of the distributed capacity of the coil they used in their experiments to be but a small fraction (2 or 3 per cent) of the total value of the capacity they used, and their coil was one of many layers in which one would expect this capacity to be large.

Since C_e , the capacity in the electrostatic system is $\frac{1}{v^2}$ times C_m , the same capacity in the electro-magnetic system, where

$$v = 3 \times 10^{10} \text{ cm sec}^{-1},$$

it follows that we may write

$$T = \frac{\lambda}{v} = 2\pi\sqrt{L_m C_m} = 2\pi\sqrt{L_m \frac{C_e}{v^2}}$$

and therefore

$$\frac{\lambda}{2} = \pi\sqrt{L_m C_e}.$$

In Drude's paper (pp. 321-323) we may obtain the value of λ' , for any coil by the following formula: $\frac{1}{2} \lambda = f \cdot l$, where l is the total

¹ Wied. Annalen, 9, p. 293.

² Trans. Camb. Phil. Soc., XVIII.

length of wire in the coil, and where f is a function to be taken from the table, and depending for a given wire, wound in a given manner, upon the ratio of the length of the coil to its diameter. Making these calculations, the following table is obtained:

TABLE V.

Showing the Approximate Values of the Distributed Capacities of the Various Sections of the Two Standard Coils.

[The values in the last column are probably correct to about 5 per cent. See also F. Dolezalek, Wied. Ann. 12, p. 1142. 1903.]

Coil	Length of coil b cm	Number turns	b/2r	f	Length of wire in cm	$\frac{1}{2} \lambda$ in cm	L_m in henrys	Distrib. Cap. in cm
1+2+3	45.84	661	0.847	1.43	1.125×10^5	1.63×10^5	0.180	14.8
1+2	32.65	472	.604	1.67	.803 "	1.34 "	.113	15.9
2+3	30.51	440	.563	1.75	.750 "	1.31 "	.102	16.9
2	17.31	251	.321	2.08	.427 "	.888 "	.0442	17.7
1	15.29	221	.282	2.12	.376 "	.800 "	.0362	17.8
3	13.15	189	.243	2.13	.322 "	.708 "	.0283	17.9
A+B	44.12	716	.814	1.47	1.240×10^5	1.80×10^5	.216	15.0
A	23.40	380	.451	1.86	.647 "	1.20 "	.0876	28.8
B	20.68	336	.382	1.95	.572 "	1.12 "	.0729	23.4

11. EFFECT OF DEVIATIONS FROM THE MEAN RADIUS.

Another point of interest in the calculation of the self-inductance of so-called circular coils with any cross section, is the influence of the deviations from circularity.

Max Wien¹ has made the following experiment. The self-inductance of a circular coil was carefully measured, the coil was then strained into an ellipse whose axes were in the ratio of two to three. The self-inductance changed from 751.5 to 730.4 arbitrary units. That such a large distortion should produce such a small variation is surprising, and proves that small variations from true circularity affect the self-inductance but little. In other words, a coil of elliptical shape, or even of an irregular shape, if the ellipticity or irregularities were small, would have almost exactly the same self-

¹ Wien Ann. 53.

inductance as a truly circular coil, as long as they had the same mean radius.

The mathematics of such a case would be formidable, but the results of the experiments of Max Wien show that it is not worth while to attempt any theoretical solution of the problem.

It can be shown also¹ that if the dimensions of the cross section of a coil are small in comparison with the radius, it makes but very little difference what the shape of the cross section is, as long as it is of constant area. That is, a coil with the same cross section has very closely the same self-inductance as a coil of the same mean radius, but whose cross section is a circle or a polygon of the same area.

Frölich² has recently constructed a standard of self-inductance consisting of a marble ring of rectangular cross section closely wound with wire. It is wound so that every turn lies in a plane passing through the axis of the ring. The self-inductance of such a coil is well known to be

$$L = 4\pi^2 a \log \frac{R+a}{R-a}$$

where $2a$ is the side of the cross section, and R is the mean radius of the ring.

It would seem that such a shape would offer great difficulties in turning accurately. It certainly requires a large number of measurements of inside and outside diameters, and of the width, both inside and outside. The wire can not possibly be equally spaced on the outside and inside, and the turning of the sharp corners, assumed in the theory, is only approximately attained in practice. The wound coil is liable to injury, as its weight, which is considerable, must always bear on the wires.

It would seem that such a coil, although possible to construct, as the fine work of Frölich has proven, is not the best form for a primary standard of self-inductance.

NOTE—In conclusion the writer wishes to acknowledge his indebtedness to Clark University for the opportunity to carry out the research and for the facilities afforded, and to Prof. A. G. Webster for his many fertile suggestions, his infinite patience, and his support and encouragement throughout the investigation.

¹ Statement of Lord Rayleigh, loc. cit. Coll. Papers, Vol. II, p. 15.

² *Frölich*: Ann. der Phy. u. Ch. 68, 1897, p. 142.

AN EFFICIENCY METER FOR ELECTRIC INCANDESCENT LAMPS.

By Edward P. Hyde and H. B. Brooks.

1. INTRODUCTION.

The life of a normal incandescent lamp depends primarily on the temperature at which the lamp is operated, and the temperature of the filament is a function, not of the watts or of the candlepower simply, but of the ratio, watts per candle; that is, of the so-called efficiency¹ at which the lamp is operated. For lamps of the same type, the higher the efficiency the shorter the life, and vice versa; and since uniformity in life is one of the first qualifications of a good lot of lamps, uniformity in efficiency is of great importance.

There is at present no direct reading "efficiency meter" on the market, so far as the writers know, and it may be that this is partly the reason for the present custom of specifying limits of watts and candlepower, and not limits of efficiency. In order to meet the need for an efficiency meter in the photometric work of the Bureau of Standards, the writers have recently designed and constructed such an instrument, the theory and operation of which will be described in this paper.

¹In the following paper the term "efficiency" will be used to designate the watts per mean horizontal candle, in accordance with the more common use of the term in industrial practice. The proper definition of the word "efficiency" would be lumens per watt, which gives a much more significant constant for a radiating source than watts per mean horizontal candle. For incandescent lamps of the same type, however, in which the ratio of mean spherical to mean horizontal intensity is approximately constant for different lamps, the watts per mean horizontal candle gives an approximate idea of the relative temperatures of the filaments. Moreover, all that is said here in regard to watts per mean horizontal candle would apply equally well to watts per mean spherical candle, if the photometer gave readings of mean spherical instead of mean horizontal intensity, and by adding a new scale to the wattmeter direct readings in lumens per watt could be made.

The problem to be solved consisted in devising some method by which the readings of a wattmeter would be made to vary inversely as the candlepower. Two general methods suggested themselves, a mechanical method and an electrical one. According to the former either the scale of the wattmeter would be moved by the motion of the photometer carriage, or the motion of the latter would vary the counterforce opposed to the moving coil, as, for example, by increasing or decreasing the effective length of the spring. Either of these methods would require the wattmeter to be specially constructed, and to be fixed in position relative to the photometer, and would present other serious practical difficulties, so that the mechanical method was abandoned for the more promising electrical method.

According to the latter there are three possible ways of decreasing the wattmeter reading in the ratio of $\frac{\text{Constant}}{\text{Candlepower}}$: (a) by placing a variable resistance in parallel with the current coil of the wattmeter; (b) by placing a variable resistance in parallel with the moving coil; and (c) by placing a variable resistance in series with the pressure circuit in the wattmeter. Before discussing these methods it should be pointed out that in every case this variable resistance must be controlled by the motion of the photometer, as by the use of a sliding contact moving over a coil, so that for all positions of the photometer the reading of the instrument shall be

$$\text{Constant} \times \frac{\text{Watts}}{\text{Candlepower}}$$

where the candlepower is given by the index on the photometer carriage. If we should make the constant equal to unity the reading on the instrument would be so small compared with the full scale deflection that only a relatively low accuracy could be obtained, so that it seemed desirable to make the constant 10, since then the reading would come at a good point on the scale and at the same time the scale reading would give the true watts per candle by moving the decimal point. To take an example, suppose that a 16-cp lamp requires 64 watts at normal voltage. In order to make the instrument read watts per candle directly we must so alter the resistance of the instrument by one of the above methods, or by

some combination of them, that the instrument shall indicate $1/16$ of its original reading; that is, $64/16=4$. If, however, instead of reducing the reading to $1/16$ we reduce it to $10/16$ of its original value, it will now give a reading of $10/16 \times 64=40$, which will be ten times the watts per candle, but which will come at a much more desirable part of the scale of the instrument. At 10 cp the instrument will read $10/10 \times$ watts, or watts directly, at 20 cp $10/20 \times$ watts or one-half the watts, and so on for other candlepower values.

To return to the three electrical methods suggested for the reduction of the wattmeter reading, case (a) is objectionable because of the low resistance of the series coil in a wattmeter of the required range. It would necessitate a low resistance in the shunt circuit controlled by the moving screen, and hence would be liable to error due to variations in the contact resistance. In case (b) this objection will not arise, since the resistance of the moving coil will be about 50 ohms, and hence the shunt controlled by the movement of the screen will not, in general, be less than 50 ohms, in comparison with which variations in the resistance of well made sliding contacts will be negligible. This method requires that an extra lead be brought out to an extra binding post on the wattmeter, as both terminals of the moving coil are not directly accessible.

In both the shunt methods the range of the efficiency meter, in case the instrument is to read 10 times the efficiency, will have as its lower limit 10 cp, for at 10 cp the resistance of the external shunt must be infinite, or, in other words, the shunt circuit must be open. Moreover, in case (b) if the instrument is to be used for candle-power values above 20 cp the shunt resistance would become smaller than 50 ohms, and hence care must be exercised to have the contact resistance sufficiently constant to produce no appreciable error. In general practice, however, in reading 16-cp lamps the range is seldom greater than from 10 to 20 cp, so that these difficulties are not serious.

Method (c) is more nearly free from errors and limitations than either of the other two methods. According to this method a variable resistance is placed in series with the pressure circuit in the wattmeter, and since the latter will have a resistance of several thousand ohms, the contact resistance is of much less importance than in the other cases. Moreover there are practically no limits to the range of the instrument, if we take out of the wattmeter part of the resist-

ance in series with the moving coil. This becomes necessary only in case we desire to use the efficiency meter below 10 cp; otherwise the wattmeter can be used without any modification. Method (c) therefore seemed more promising than either of the other two methods, and was hence adopted.

2. THEORY OF METHOD.

Assuming that the wattmeter to be used reads correctly as a wattmeter, the resistance in series with it should be zero at 10 cp, since the reading at 10 cp should be total watts. At 20 cp the reading should be one-half the total watts; that is, the added resistance for the position of the photometer corresponding to 20 cp should equal the resistance of the wattmeter. It only remains, therefore, to determine the relation between the candle-power reading and the resistance for other points on the bar, and to find some convenient practical way of introducing the additional resistance.

Let $2a$ be the distance (supposed to be constant) between the two lamps, I_1 the intensity of the comparison lamp, I the intensity of the test lamp, and x the distance from the test lamp to the photometer screen when in a position of balance. Then by the inverse square law,

$$I = I_1 \left(\frac{x}{2a - x} \right)^2 \quad (1)$$

Moreover, if W is the total watts supplied to the lamp, and E the efficiency or watts per candle,

$$E = \frac{W}{I} \quad (2)$$

and if S denote the scale reading of the instrument when the external resistance r is in series with the wattmeter pressure winding of resistance r_1 , and K the factor of proportionality between the scale reading and the efficiency, we get

$$S = \frac{r_1}{r_1 + r} W = KE = K \frac{W}{I} \quad (3)$$

or

$$\frac{r_1}{r_1 + r} = \frac{K}{I} \quad (4)$$

from which

$$r = r_1 \left(\frac{I}{K} - 1 \right) \quad (5)$$

Substituting for I its value as given in equation (1),

$$r = r_1 \left[\frac{I_1}{K} \left(\frac{x}{2a-x} \right)^2 - 1 \right] \quad (6)$$

which gives the resistance r , which must be added to the resistance r_1 for any position x on the bar.

The method which seemed most desirable for the construction of a rheostat to follow this law consisted in winding closely a fine silk-covered resistance wire on a block of insulating material. After coating with shellac varnish and drying, the insulation was removed along a strip over which travels a contact brush attached to the photometer carriage. The variable rheostat thus formed is connected in the pressure circuit of the wattmeter, so that the resistance of this circuit varies with the position of the photometer. The electrical connections are shown diagrammatically in Fig. 1. The top of the block, along which the contact is made, is straight, and the bottom of the block must have whatever form is necessary to give the proper increments dr to the resistance as the contact moves over successive small distances dx . By differentiating equation (6) with respect to x we get the equation,

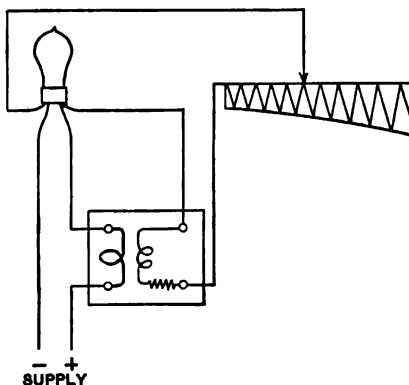


Fig. 1.—Diagram of Connections.

$$\frac{dr}{dx} = \frac{4ar_1I_1}{K} \frac{x}{(2a-x)^3} \quad (7)$$

If we let h be the thickness of the block, y the height of the block at any point—that is, the ordinate of the curve measured downward— ρ the specific resistance of the wire, $2b$ the diameter of the bare wire,

and n the number of turns of the covered wire per unit distance on the block, then the resistance dr contained in a small distance dx is

$$dr = \frac{2(y+h)\rho}{\pi b^3} \times ndx \quad (8)$$

or

$$\frac{dr}{dx} = \frac{2n\rho}{\pi b^3}(y+h) \quad (9)$$

Equating this value of $\frac{dr}{dx}$ to that obtained in equation (7)

$$\frac{2n\rho}{\pi b^3}(y+h) = \frac{4ar_1I_1}{K} \frac{x}{(2a-x)^3} \quad (10)$$

from which

$$y = \frac{2\pi ab^3r_1I_1}{Kn\rho} \frac{x}{(2a-x)^3} - h, \quad (11)$$

which relation determines the form of the bottom of the block. In this equation a , r_1 , I_1 , K , and h may be taken as constants, while b , n , and ρ depend on the wire that is used. We may introduce the resistance per unit length, ρ_1 , instead of the specific resistance ρ , using the relation

$$\rho_1 = \frac{\rho}{\pi b^3}$$

Combining this with equation (11) we get the more convenient form

$$y = \frac{2ar_1I_1}{Kn\rho_1} \frac{x}{(2a-x)^3} - h, \quad (12)$$

involving the two wire constants n and ρ_1 . If then a block of insulating material is constructed, having the general form indicated in Fig. 2, the curvature of the bottom conforming to equation (12), and if the wire whose constants are used in equation (12) is wound closely on this block, the variable rheostat thus formed, if put in the pressure circuit of the wattmeter, as indicated in Fig. 1, will cut down the reading of the wattmeter for all positions of the photometer within the limits of the coil, so that the instrument will indicate, not watts, but a constant K times the efficiency of the lamp.

3. CONSTRUCTION OF INSTRUMENT.

It has been shown that it is desirable to give the constant K the value 10, since then the wattmeter reads watts per candle directly, by moving the decimal point. We will assume that the apparatus is to be designed for 16-cp lamps. In adjusting a photometer for the measurement of 16-cp lamps the voltage on the comparison lamp is adjusted until the reading on a 16-cp standard lamp placed in the test-lamp socket comes at 16 cp, the middle point on the bar. Then the intensity of the comparison lamp, I_1 , is 16 cp. The distance between the two lamps in commercial photometry is always constant and equal to some standard length, such as 100 or 200 inches, or 250 cm. Hence a , which is one-half this distance, is the constant distance from either lamp to the center of the bar. In the commercial photometer at the Bureau of Standards $a = 125$ cm. A Weston wattmeter was used having a resistance of 2,630 ohms. The other constants entering into equation (12) are h , the thickness of the block, which was made 1 cm, and the two wire constants n and ρ_1 . In designing the resistance block it is necessary to work both ways from equation (12). First having decided what the approximate dimensions of the block shall be, put in equation (12) the value of y for some value of x , preferably that corresponding to 20 cp, and from this equation determine the product $n\rho_1$. Then select some wire having approximately this value of $n\rho_1$, put the exact values, as obtained from measurements of a sample of the wire, back into equation (12), and plot the desired curve.

It may be well to say a word at this point in regard to the limits of the efficiency meter in practice. As stated, if we make $K = 10$ there will be no resistance in series with the wattmeter when the photometer is at 10 cp. Hence the coil begins at this point unless it is desired to read efficiencies for points below 10 cp, in which case some of the resistance in series with the moving coil can be taken out of the wattmeter and placed in the external coil. This can be accomplished without unfitting the wattmeter for other uses, by bringing to an extra binding post on the instrument a tap from some point in the resistance depending on how far below 10 cp the meter is to be used. While there is no difficulty in doing this, there is also no need generally of having the meter read below 10 cp on a 16

cp bench. The upper limit can be as high as desired, although the ordinates of the block increase very rapidly as we reach higher candle-power positions, so that unless there is some particular reason to the contrary it will be found desirable to take as the upper limit 20 or 25 cp. The meter at the Bureau reads watts per candle for a range of from 10 to 20 cp.

We have spoken thus far of a single block wound with wire; we have found it very desirable, however, to have two blocks instead of one. First, for any wire large enough for mechanical strength the slope of the curve at 20 cp is so great that it not only makes an awkward-looking device, but also introduces difficulties in the winding. By using two blocks side by side, equal in every way, the curvature of each is much less than that of the single block. Secondly, with the single block the sliding contact that moves with the photometer must carry with it a flexible wire, which is objectionable. If two blocks are used, however, the photometer merely carries a brush which spans the gap between the two coils. In this arrangement the ends of the wire at the 20-cp point are joined together, keeping the wattmeter circuit closed in case the contact between the brush and the coil should be momentarily broken.

For two blocks of the same thickness the values of $y+h$ deduced from equation (12) must be divided by 2, so that

$$y+h = \frac{1}{2} \times \frac{2ar_1 I_1}{Kn\rho_1} \frac{x}{(2a-x)^3}$$

or

$$y = \frac{ar_1 I_1}{Kn\rho_1} \frac{x}{(2a-x)^3} - h \quad (13)$$

If we put

$$C = \frac{ar_1 I_1}{Kn\rho_1} \quad (14)$$

we get

$$y = C \frac{x}{(2a-x)^3} - h \quad (15)$$

where C is a constant depending on the values chosen for n and ρ_1 . "Climax" wire was used on account of its high specific resistance, which allows the use of the largest possible wire for a given size of block. The size of the wire was No. 30 B. & S., and n and ρ_1 were

33 turns per cm, and 0.1623 ohm per cm, respectively. Substituting the values in equation (14), and putting for the other constants the values given above,

$$\begin{aligned}K &= 10 \\a &= 125 \\r_1 &= 2630 \\I_1 &= 16\end{aligned}$$

we get

$$\begin{aligned}C &= \frac{125 \times 2630 \times 16}{10 \times 33 \times 0.1623} \\&= 9.82 \times 10^4\end{aligned}$$

Putting this value for C in equation (14),

$$y = 9.82 \times 10^4 \times \frac{x}{(250-x)^3} - h. \quad (16)$$

As the edges of the block must be rounded for better winding, the effective value of h is slightly reduced, and a suitable allowance must be made. In equation (16) x is the distance along the bar measured from the test lamp, and is determined for any candle-power, I , by the equation

$$\frac{I}{I_1} = \frac{I}{16} = \frac{x^3}{(250-x)^3} \quad (17)$$

from which

$$x = \frac{250\sqrt[3]{I}}{\sqrt[3]{I}+4} \quad (18)$$

From equations (18) and (16) we get the numerical values:

$$\begin{array}{lll}I = 10 \text{ cp} & x = 110.38 \text{ cm} & y = 3.08 \text{ cm} \\I = 16 \text{ cp} & x = 125.00 \text{ cm} & y = 5.38 \text{ cm} \\I = 20 \text{ cp} & x = 131.97 \text{ cm} & y = 6.98 \text{ cm}\end{array}$$

The complete curve from 10 cp to 20 cp may be seen in Fig. 2, which gives the form in which the blocks should be made. We used for the blocks an insulating material called "stabilit," made by the Allgemeine Electricitäts Gesellschaft of Berlin. This material

is said to possess high insulation resistance combined with the ability to resist heat and moisture. It was used in preference to fiber because of the hygroscopic nature of the latter. Hard rubber would no doubt answer the purpose as well as stabilit.

Lack of uniformity in the wire or in winding may produce an accumulating error, so that it is desirable to test the resistance at a number of points as the wire is wound. In this connection it may be well to call attention to another way of considering the curve in Fig. 2, which is helpful in checking the resistance from point to point. Since the reading at 11 cp is to be $\frac{1}{2}$ of the reading at 10 cp, the total resistance of the pressure circuit at this point should be $\frac{1}{2}$ of the resistance of the wattmeter, so that the added resistance should be $\frac{1}{2}$ the resistance of the wattmeter. Similarly the added resistance between 11 cp and 12 cp, or between 12 cp and 13 cp, should be $\frac{1}{2}$ the resistance of the wattmeter. Hence the resistance between points on each block corresponding to successive candlepowers should be equal to each other and equal to $\frac{1}{2}$ the resistance of the wattmeter. If greater accuracy is desired the resistance may be checked during the winding at every half candlepower point. Moreover, if slight errors are introduced in winding one coil, they may be partially neutralized in winding the other coil, so that the resultant error will be very small.

The wattmeter should be calibrated to read correctly as a wattmeter over the range for which it will be used, in order to avoid the necessity for applying a correction to the efficiency readings. It should have a compensating winding on the fixed coil, through which the potential current is carried in the opposite direction to the main current, so that the energy used in the potential circuit will not be measured along with that supplied to the lamp. This compensating winding is regularly supplied in low-range wattmeters by at least one well-known maker.

The reading of a wattmeter on direct current is affected by the earth's field. To eliminate the error from this source it is customary to take the mean of two readings, the direction of current flow through the circuits of the instrument being reversed for the second reading. Another way of accomplishing the same result is to balance the earth's field at the moving coil by the use of a small permanent magnet. This is the method we have used. A magnet

10 cm long and 1.3 cm diameter, when placed north and south at about five or six cm under the bottom of the wattmeter case balanced the earth's field almost exactly for all points of the scale that are used (see below), the wattmeter being placed with its longer dimension approximately north and south. Another way of eliminating the effect of the earth's field is to use an astatic wattmeter. Such instruments are not on the market in portable form, but they can doubtless be had from instrument makers if required. If the instrument is to be used in a location subject to strong and variable stray fields the use of an astatic wattmeter might be necessary, but in most cases the ordinary wattmeter with a compensating magnet will doubtless answer all requirements.

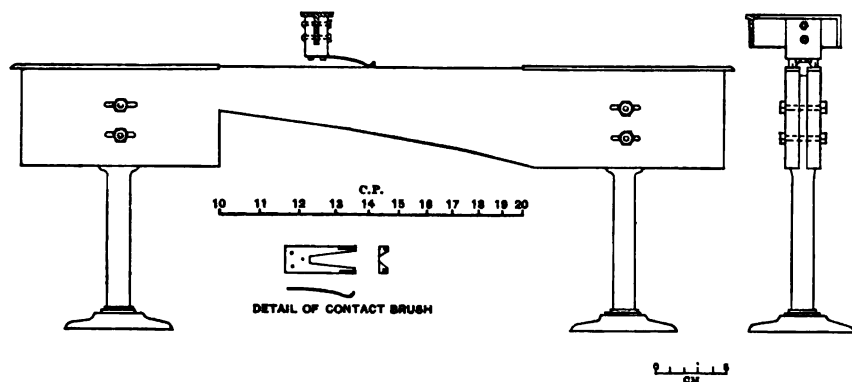


Fig. 2.—*Blocks and Supports for Variable Rheostat.*

Fig. 2 is the working drawing from which the blocks and supports were made. Although a range of from 10 to 20 cp is sufficient, it is desirable to make the blocks longer, so that the wattmeter circuit will not be broken should the photometer move beyond these limits. On the top of each block, beyond the limits of the coil at each end, a copper strip was placed over which the contact brush moves when off the coil. The resistance wire at the end of the coil corresponding to 10 cp was soldered directly into the copper strip at that end, but the other end of each coil was insulated from the metal strip beyond it. On each block the copper strip at one end was joined to the copper strip at the other end, so that the instrument reads watts per candle so long as the contact brush remains on the coil—that is, for all candlepower values between 10 and 20, while

beyond these limits the instrument reads watts. In order that for any position of the photometer the watts can be read directly, a switch is used by means of which the resistance external to the wattmeter can be short-circuited, so that either watts or efficiency can be read.

The blocks were mounted on stands with the tops of the coils at a convenient distance below the photometer carriage. On the latter was mounted a hard-rubber block carrying the phosphor bronze contact brush which completes the circuit from one coil to the other.

The blocks were slotted to receive bolts holding them in place on the stands. Thus a linear adjustment of each coil independently can be made, which is desirable. Fig. 3, which was made from a photograph, shows the blocks in position on the photometer, the wattmeter not being shown.

4. EXPERIMENTAL RESULTS.

The coils having been mounted and the wattmeter put in place, it was necessary to find the position of the small magnet in which it would neutralize the earth's field at the moving coil for the range used. By trial, using a 16-cp lamp, a position was found such that the wattmeter reading at one point was the mean of direct and reversed readings taken before the magnet was put in place. To test other points, the photometer was then set at 10, 16, and 20 cp, direct and reversed readings being made at each point. The agreement of the results is shown in Table I.

TABLE I.

Photometer position	Wattmeter reading	
	Direct	Reversed
10 cp	49.3	49.4
16	30.8	30.8
20	24.6	24.6

The next step consisted in placing the blocks in proper relation to the sliding contact to give a correct reading of watts per candle. To do this a 16-cp lamp was placed in the socket of the rotator, and

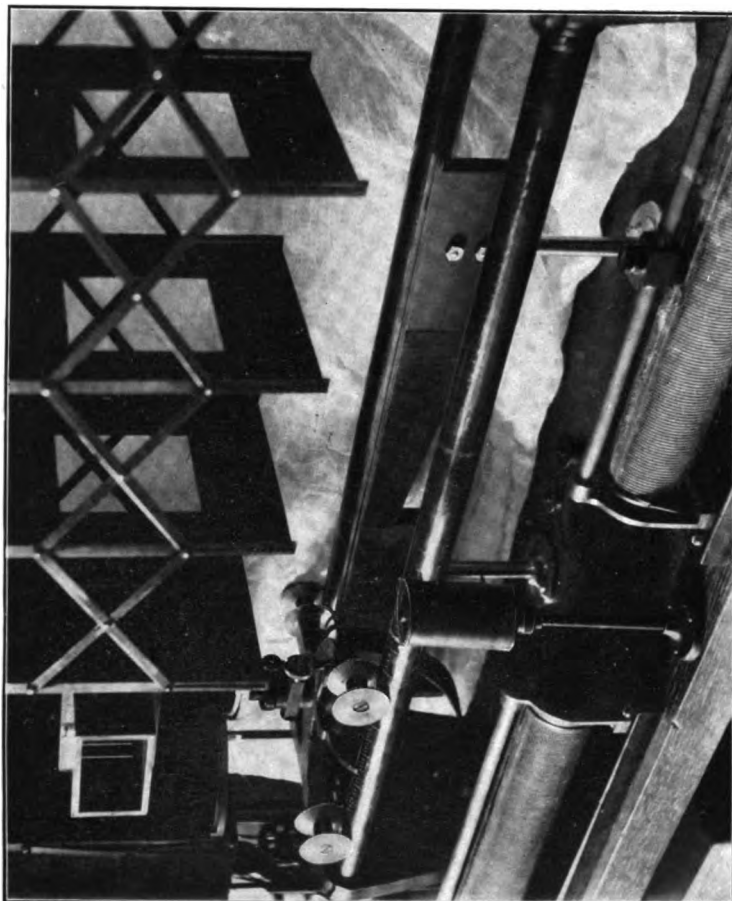


Fig. 3.— Variable Rheostat in Position on Photometer.



the screen was set at 16 cp. The blocks were then adjusted until the wattmeter read the known efficiency of the lamp, and were then bolted to the supports. The efficiency meter was then in calibration for this one point. To check it at other points the lamp was kept at the original voltage, the photometer screen was placed successively at different positions on the bar, and readings were taken. The accuracy of the instrument at all points is shown in Table II, in which the first column contains the successive positions of the photometer on the bar. The second column gives the calculated efficiencies corresponding to these positions, as determined by dividing the watts supplied to the lamp by the values given in the first column. The third column contains the corresponding values observed on the efficiency meter.

TABLE II.

Photometer position	Calculated efficiency	Observed efficiency
10 cp	4.78	4.78
11	4.34	4.34
12	3.98	3.98
13	3.68	3.67
14	3.41	3.40
15	3.18	3.18
16	2.98	2.98
17	2.81	2.80
18	2.65	2.63
19	2.51	2.51
19.8	2.41	2.40

It is thus seen that for 16-cp lamps ranging in efficiency from 2.4 to 4.8 watts per candle the apparatus gives true results to within less than 1 per cent. The instrument as adjusted for 16-cp lamps can be used without change for 32-cp and 8-cp lamps. If we place a 32-cp lamp in the comparison lamp socket, so that the photometer comes in the middle of the bar for 32-cp test lamps, the efficiency of such lamps can be read by dividing the reading of the instrument by 20 instead of by 10. In our instrument the wattmeter has two scales, the upper with a range of 0-150 watts, being used for 16-cp

lamps. By using the lower scale of 0-75 watts when measuring 32-cp lamps the instrument is still direct reading. This refers only to the reading of the deflection; the electrical connections are the same for 16 and for 32-cp lamps.

The accuracy of the instrument when used for 32-cp lamps depends on two things; first, whether the wattmeter is as accurate at the higher readings as at the ones used in measuring 16-cp lamps, and secondly, whether the earth's field is still balanced at the moving coil for the larger deflections. To test this point the observations given in Table III were made. A 32-cp lamp was placed in the test lamp socket and readings were made with the connections direct and reversed for different positions of the photometer, similar to the readings given in Table I for 16-cp lamps. The position of the small magnet was the same as before. The compensation for the earth's field is seen to be practically perfect.

TABLE III.

Photometer position	Wattmeter reading	
	Direct	Reversed
20 cp	113.9	113.9
32	70.8	70.8
39.6	57.2	57.3

In a similar way the same instrument can be used for 8-cp lamps by dividing the wattmeter readings by 5. An objection to this is that the deflection being about one-half that for 16-cp lamps, the accuracy of reading is reduced. It is still, however, ample for commercial testing.

5. APPLICATIONS.

Besides the ordinary use of the instrument in reading watts per candle when lamps are tested for candlepower at some constant voltage, or for voltage at some fixed candlepower, it is very convenient for determining directly the voltage at which a lamp has a definite efficiency. Oftentimes, as in making life tests, it is desirable to know at what voltage the lamps have a given efficiency, say 3.1

watts per candle. The only way to do this at present is to estimate the approximate voltage, and from the candlepower and wattage readings at that voltage to calculate the voltage corresponding to the required efficiency. This computation involves the use of the law of variation of efficiency with voltage, and is hence both tedious and unreliable. By the use of the efficiency meter, on the other hand, one operator gradually raises the voltage and observes the efficiency meter, while the other observer follows the change in candlepower with the photometer. When the instrument indicates the proper efficiency, the corresponding voltage is read. This method is not only quicker, but much more accurate and satisfactory.

If the utmost accuracy is desired, certain precautions must be taken. If the line voltage is considerably higher than the voltage at the lamp—that is, if there is a large drop through the rheostats—any change in the current will cause a relatively large change in the voltage on the lamp. Hence if the voltage at the lamp is adjusted when the photometer carriage is at 10 cp, the voltmeter will show a slight rise when the photometer is moved to 20 cp, due to the increase of resistance of the pressure circuit, and the consequent decrease in the pressure current. With the extreme travel of the carriage just given, this rise would be a few tenths of a volt, if the drop in the rheostats is 10 volts and 16-cp lamps are being tested. In practice, however, there is almost never any occasion to move the photometer over more than three or four candlepower after the voltage has been adjusted, and moreover the effect of a motion of three or four candlepower at 16 cp is not so great as at 10 cp, because of the higher resistance in the pressure circuit. A motion of the photometer from 14 to 18 cp would cause a rise in the voltage of about one-tenth of a volt, with 10 volts drop through the rheostats. In the second place, the observer at the photometer would bring the screen to a position of approximate balance immediately and the second observer could correct for any small initial change, such initial change being rather common for other reasons. Finally, if the line voltage is kept but slightly above the lamp voltage, as may be done when a storage battery is used, the error becomes entirely negligible in ordinary cases. If the tests are made on a generator circuit where the voltage is unsteady it will usually be found desirable to use the “same circuit” method, in which case both lamps are changed about the same by

the motion of the photometer, so that the error is negligible. This is the only source of error peculiar to the efficiency meter. The other sources of error are those which must be guarded against in any case. For example, whether we are measuring the current on an ammeter, the watts on a wattmeter, or the watts per candle on an efficiency meter, we must guard against errors due to the voltmeter current. If we read with the voltmeter in circuit we must correct the ammeter or wattmeter reading. In the case of the efficiency meter, either a correction can be applied, as for the other two instruments, or, if the meter is to be used for testing some one type of lamp, as 16-cp, 110-volt, 3.5-watt-per-candle lamps, the coils can be moved slightly on the supports until the instrument reads correctly at 16 cp with the voltmeter in circuit. The differential errors at other points on the scale will be negligibly small in commercial testing, and the efficiency of a lamp under test may be read at once, no corrections being necessary. This gives the efficiency meter an advantage over the ammeter or wattmeter as ordinarily used.

If we should set the voltage and then open the voltmeter switch, the voltage on the lamp would rise if there is any drop through the rheostats, so that the error in the efficiency, due to the change in current in the current circuit, would be much less important than the error in the intensity of the lamp, due to the rise in voltage. If the drop through the rheostats is small, so that the photometer error, due to opening the voltmeter switch, becomes negligible, the error in the efficiency can be eliminated by adjusting the coils with the voltmeter switch open, and then always reading the efficiency under the same conditions. It is necessary to make readings under the same conditions in any case, for even if the photometer error is eliminated by having the line voltage approximately equal to the lamp voltage, the reading of the ammeter or the wattmeter depends on whether the voltmeter is thrown on the test lamp or on the comparison lamp. Hence, to avoid errors, a routine must be followed, allowing proper corrections in case the voltmeter is left on the test lamp while the ammeter or the wattmeter is read.

We thus see that the efficiency meter, while subject to no greater errors than the ammeter or the wattmeter, is more flexible than either, and gives by a direct reading one of the most important constants of the incandescent lamp.



CALCULATION OF THE SELF-INDUCTANCE OF SINGLE-LAYER COILS.

By Edward B. Rosa.

1. THE FORMULÆ.

A well-constructed standard of self-inductance may be measured with considerable precision in terms of a resistance. If its value be computed from its dimensions, the two results should agree, provided the resistance is known in absolute measure. This affords a method of determining resistance in absolute measure, the success of which depends, of course, on how accurately the self-inductance can be computed, as well as upon the accuracy of the measurements.

Kirchhoff's method of determining resistance in absolute measure consists essentially in finding by experiment the mutual inductance of a pair of coils in terms of a certain resistance, the value of the mutual inductance having been calculated from the dimensions of the coils and their distance apart. Placing these two values equal to each other, the value of the given resistance becomes known in absolute measure. The mutual inductance of such a pair of coils can be determined experimentally by two measurements of self-inductance, in one of which the current flows in the same direction in the two coils and in the other it flows in opposite directions. If L and L' are the two values of self-inductance determined experimentally, L_1 and L_2 the self-inductances of the two coils separately, and M their mutual inductance, then

$$\begin{aligned}L &= L_1 + 2M + L_2 \\L' &= L_1 - 2M + L_2 \\ \therefore M &= \frac{L - L'}{4}\end{aligned}$$

Such a pair of coils may have a large radius and relatively small rectangular cross section, such as used by Rowland and Glazebrook, or they may be coaxial solenoids of equal length, or one may be long and the other quite short, the short one being either within or without the long one. All these cases may be calculated quite accurately when the dimensions are accurately known.

The simplest method, however, is to use a single coil of relatively large dimensions, wound with a single layer of wire, (1) calculating its self-inductance from its dimensions and (2) measuring its self-inductance directly in terms of a resistance. The dimensions of such a coil may be measured with great precision, supposing it wound on an accurately ground marble cylinder. In an article in the last number of this Bulletin Professor Coffin¹ gave a description of such a standard of inductance, and calculated its value by two

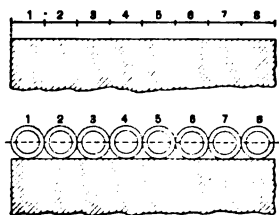


Fig. 1.

different formulæ, one being an absolute formula using elliptic integrals, first given by Lorenz,² and the second a formula derived by Coffin in the form of a converging series, being an extension of Rayleigh's formula for a cylindrical winding of a single layer. These two formulæ give results agreeing to within 1 part in 50,000, an agreement which is highly satisfactory.

As the measurements of the coil may readily be made of sufficient accuracy, the only question remaining is whether the two closely agreeing formulæ are as accurate when applied to the coil in question as they appear to be.

In deriving the formulæ the current is supposed to be uniformly distributed over the surface of the cylinder, whereas the coil is actually wound with 661 turns of round wire, having a diameter of 0.0634 cm and insulated by a covering 0.0030 cm thick. Instead of the single-current sheet *ab* first assumed in deriving the formulæ, the current sheet is subsequently supposed to be divided into n sections having altogether a self-inductance n^2 times as great as the single current sheet. This can be realized if we assume a winding of n turns of a flat strip, l/n wide (where l is the length of the wind-

¹ No. 4, March, 1906.² Wied. Annalen, 7, p. 170; 1879.

ing), and no thickness, wound uniformly over the cylinder, at a distance of $\frac{l}{2n}$ cm from the surface of the cylinder. If the edges of the strip come together but do not make electrical contact, such a winding would be equivalent to a uniform current sheet and the formulæ for the latter would apply if n^2 is inserted as a factor in L , n being the number of turns. In the case of the coil constructed for the Bureau of Standards, when $n=661$, the wire is so small and the number of turns so large it was assumed by Coffin that the actual winding is substantially equivalent to the current sheet to which the formulæ strictly apply.

2. CASE OF SHORT COIL.

Let us first examine the case of a short coil having a radius of 25 cm and a length of 1 cm, wound with 10 turns of wire, the bare wire being 0.08 cm and the covered wire 0.10 cm in diameter. Coffin's formula for a short single-layer coil of radius a and length b reduces in this case to

$$L = 4\pi n^2 a \left\{ \left(1 + \frac{b^2}{32a^2} \right) \log \frac{8a}{b} + \frac{b^2}{128a^2} - 0.5 \right\} \quad (1)$$

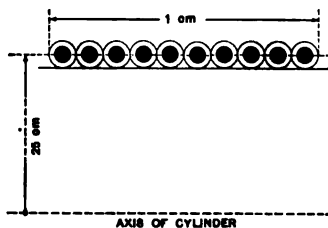


Fig. 2.

the terms neglected amounting to less than one part in a hundred million of L . This is the formula given by Rayleigh³ in 1881.

Substituting $a=25$, $b=1$, $n=10$, we have for the self-inductance

$$L = 4\pi a \times 100 \left\{ \left(1 + \frac{1}{32 \times 625} \right) \log_8 200 + \frac{1}{128 \times 625} - 0.5 \right\}$$

$$\text{or, } L = 4\pi a \times 479.8595 \text{ cm.}$$

This value assumes the current flowing in a sheet, as though the coil were wound with 10 turns of very thin tape 1 mm wide, the radius of the winding being 25 cm.

³Proc. Roy. Soc., 82, 1881, and Collected Papers.

The summation formula for the self-inductance of a single-layer coil of n turns is as follows:

$$L = nL_1 + 2(n-1)M_{12} + 2(n-2)M_{13} + 2(n-3)M_{14} + \dots + 2M_{1n} \quad (2)$$

where L_1 is the self-inductance of a single turn, M_{12} is the mutual inductance of any two adjacent turns, M_{13} is the mutual inductance of the first and third or any two turns separated by one, and M_{1n} is the mutual inductance of the first and last turns. For a coil of 10 turns this becomes

$$L = 10L_1 + 18M_{12} + 16M_{13} + 14M_{14} + \dots + 2M_{110} \quad (3)$$

If we calculate L_1 and the nine M s, we can compute the self-inductance of the actual coil wound with wire of any particular size, the only assumption being that the current is uniformly distributed over the cross section of the wire. Max Wien's formula⁴ for the self-inductance of a single circular turn of wire of radius a and radius of cross section ρ is

$$L = 4\pi a \left\{ \left(1 + \frac{\rho^2}{8a^2} \right) \log \frac{8a}{\rho} - 1.75 - .0083 \frac{\rho^2}{a^2} \right\} \quad (4)$$

Substituting $a = 25$, $\rho = .04$, we have

$$\begin{aligned} L_1 &= 4\pi a \left\{ \left(1 + \frac{2}{(2500)^2} \right) \log .5000 - 1.75 - .0083 \frac{16}{(2500)^2} \right\} \\ &= 4\pi a \times 6.76720 \text{ cm.} \end{aligned}$$

Maxwell's formula for the mutual inductance of two coaxial circles near each other reduces to the following when the two radii are equal, a being the common radius and b the distance between the circles.

$$M = 4\pi a \left\{ \left(1 + \frac{3}{16} \frac{b^2}{a^2} \right) \log \frac{8a}{b} - 2.0 - \frac{1}{16} \frac{b^2}{a^2} \right\} \quad (5)$$

$$\begin{aligned} \text{Hence, } M_{12} &= 4\pi a \left\{ \left(1 + \frac{3}{16} \frac{.01}{625} \right) \log .2000 - 2 - \frac{1}{16} \frac{.01}{625} \right\} \\ &= 4\pi a \times 5.600924 \text{ cm.} \end{aligned}$$

⁴ Wied. Annalen, 58, p. 934; 1894.

Giving b the successive values 0.2, 0.3, 0.4, etc., the other values of the M s are derived. The following values of the 10 terms of the summation formula (3) are thus obtained:

10 $L_1 = 67.6720$	8 $M_{17} = 30.4779$
18 $M_{18} = 100.8166$	6 $M_{18} = 21.9346$
16 $M_{18} = 78.5254$	4 $M_{19} = 14.0898$
14 $M_{14} = 63.0344$	2 $M_{110} = 6.8099$
12 $M_{15} = 50.5787$	
10 $M_{16} = 39.9189$	Sum $= 473.8582 = \frac{L}{4\pi a}$

This is less than the value found for $\frac{L}{4\pi a}$ by formula (1) by 6.0013, which is about 1.25 per cent. It will be noticed that the values of the mutual inductances are independent of the size of the wire with which the coil is wound, but that the self-inductance is not. Thus, if the wire were only half a millimeter in diameter, 10 L_1 would be 79.3034. In the latter case the total would be 5.6139 *more* by the summation formula than by the current sheet formula; that is, it would be more than 1 per cent greater instead of being 1.25 per cent smaller, as when the wire is 0.08 cm in diameter. It is evident that the summation formula, which takes account of the actual size and position of the wires on the coil, gives the true values and that the current sheet formula, which is very exact for a current sheet or for a winding of thin strip which is equivalent to a current sheet, can not be applied without modification to a winding of round wires. Recognizing the fact that the two cases are not identical, Coffin applies a correction by reducing the length of the solenoid, supposing the equivalent current sheet to extend only to the centers of gravity of the outer semicircles of the end wires; that is, from a to b , Fig. 3. In this case the length of the winding would be shortened by 0.046 cm, making it 0.954 cm instead of 1.0 cm. Using this value of b in formula (1), we find the self-inductance of the coil to be

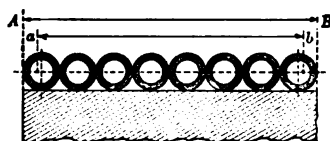


Fig 3.

$$L = 4\pi a \times 100 \left[\left(1 + \frac{(.954)^4}{20,000} \right) \log_e \frac{200}{.954} + \frac{(.954)^4}{80,000} - 0.5 \right] \quad (6)$$

$$= 4\pi a \times 484.567$$

This is 4.707 (about 1 per cent) more than the former value, which is already too great. Thus, the correction makes the matter worse. It is evident that a series of round wires can not be regarded as equivalent to a current sheet, no matter how fine the wires are; for the magnetic field is very intense close to the wires, and the smaller the wire the stronger is the field and the greater is the self-inductance. For the case of mutual inductance between two coils at a distance from each other, the size of the wires is, of course, immaterial (if not too great), and hence single-layer coils, so far as mutual inductance is concerned, are equivalent to current sheets.

3. SELF-INDUCTANCE OF SINGLE TURNS OF THIN STRIP.

In order properly to compare the results given by the summation formula with the formula for the current sheet, we may apply the former to the case of a winding of very thin tape, to which the latter applies strictly. Let the tape be 1 mm wide, of infinitesimal thickness, and let 10 turns be applied to the cylinder of radius 25 cm. The length b is then 1 cm, and the value found above by formula (1) is accurate, namely,

$$\frac{L}{4\pi a} = 479.8595.$$

To find $10 L_1$, the first term of the summation formula, we use equation (5), replacing M by L and b by R , the geometric mean distance of the strip from itself; for the self-inductance of a conductor is equal to the mutual inductance of two filaments having a distance apart equal to the geometric mean distance of the conductor from itself. The g. m. d. R of a flat strip of negligible thickness is 0.223130 of its breadth.⁵ Hence

$$\begin{aligned} L_1 &= 4\pi a \left[\left(1 + \frac{3}{16} \frac{R^2}{a^2} \right) \log \frac{8a}{R} - 2 - \frac{1}{16} \frac{R^2}{a^2} \right] \\ &= 4\pi a \left[\log \frac{8a}{R} - 2 \right], \end{aligned} \quad (7)$$

neglecting terms in $\frac{R^2}{a^2}$, which here amount to less than one part in a million.

⁵Maxwell II, § 692.

Here $a = 25$ cm and $R = .0223130$ cm. Substituting these numerical values, we find

$$\frac{10 L_1}{4\pi a} = 71.0090.$$

4. THE GEOMETRIC MEAN DISTANCE OF ONE STRIP FROM ANOTHER.

In order to calculate the mutual inductance of the several strips upon one another it is necessary to know their geometric mean distance from one another. Let Fig. 4 represent the section of the first and third turns of the winding, having a width a . Find first the g. m. d. of a point P distant b from the end taken as origin, from the line CD. Then

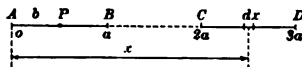


Fig. 4.

$$\begin{aligned} a \log R &= \int_a^{3a} \log (x-b) dx = \left[(x-b) \log (x-b) - (x-b) \right]_a^{3a} \\ &= (3a-b) \log (3a-b) - (2a-b) \log (2a-b) - a \end{aligned} \quad (8)$$

where R is the g. m. d. of the point P from the line CD. To find the g. m. d. of all points in AB from the line CD we integrate again, this time along the line AB, first changing b in (8) to x . Thus,

$$\begin{aligned} a^2 \log R_1 &= \int_0^a (3a-x) \log (3a-x) dx - \int_0^a (2a-x) \log (2a-x) dx \\ &\quad - \int_0^a a dx \\ &= 9 \frac{a^2}{2} \log 3a - 4a^2 \log 2a + \frac{a^2}{2} \log a - 3 \frac{a^2}{2} \\ \text{or } \log R_1 &= \frac{9}{2} \log 3a - 4 \log 2a + \frac{1}{2} \log a - \frac{3}{2} \end{aligned} \quad (9)$$

If we consider the first and fourth strips, the limits of integration would be $3a$ and $4a$, and 0 and a , respectively, and we should get

$$\log R_2 = \frac{16}{2} \log 4a - 9 \log 3a + \frac{4}{2} \log 2a - \frac{3}{2} \quad (10)$$

The general expression is

$$\log R_n = \frac{(n+1)^2}{2} \log (n+1)a - n^2 \log na + \frac{(n-1)^2}{2} \log (n-1)a - \frac{3}{2} \quad (11)$$

where na is the distance from the center of one element to the center of the other, a being the breadth of each element. Making n in the general expression equal to 0, 1, 2, 3, 4, etc., successively, we may find the values of the geometric mean distances of the first strip from the other nine.

The above formula is, however, not well adapted to numerical calculation when n is large, as the logarithm of R is the difference between large positive and negative terms, and unless the latter are calculated with extreme accuracy there is likely to be an appreciable error in the differences. The expression for $\log R$ may, however, be transformed into a series well adapted to numerical calculation for all values of n greater than unity.

Expanding the coefficients of equation (11) and recombining the terms we have, putting a equal to unity,

$$\begin{aligned} \log R_n &= \frac{n^2}{2} \log (n^2 - 1) - \frac{n^2}{2} \log n^2 + n \log \frac{n+1}{n-1} + \frac{1}{2} \log (n^2 - 1) - \frac{3}{2} \\ &= \frac{n^2}{2} \log \left(\frac{n^2 - 1}{n^2} \right) + \frac{1}{2} \log \left(\frac{n^2 - 1}{n^2} \right) + \frac{1}{2} \log n^2 + n \log \frac{n+1}{n-1} - \frac{3}{2} \\ &= \frac{n^2 + 1}{2} \log \left(1 - \frac{1}{n^2} \right) + n \log \left(\frac{n+1}{n} \right) - n \log \left(\frac{n-1}{n} \right) + \log n - \frac{3}{2} \quad (12) \end{aligned}$$

Expanding the first three terms on the right of equation (12),

$$\begin{aligned} \log R_n &= \log n - \frac{3}{2} - \frac{n^2 + 1}{2} \left(\frac{1}{n^2} + \frac{1}{2n^4} + \frac{1}{3n^6} + \frac{1}{4n^8} + \dots \right) \\ &\quad + 2n \left(\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \frac{1}{7n^7} + \dots \right) \\ &= \log n - \frac{3}{2} - \left(\frac{1}{2} + \frac{1}{4n^2} + \frac{1}{6n^4} + \frac{1}{8n^6} + \frac{1}{10n^8} + \dots \right) \\ &\quad - \left(+ \frac{1}{2n^2} + \frac{1}{4n^4} + \frac{1}{6n^6} + \frac{1}{8n^8} + \dots \right) \\ &\quad + \left(2 + \frac{2}{3n^2} + \frac{2}{5n^4} + \frac{2}{7n^6} + \frac{2}{9n^8} + \dots \right) \\ \therefore \log R_n &= \log n - \left(\frac{1}{12n^2} + \frac{1}{60n^4} + \frac{1}{168n^6} + \frac{1}{360n^8} + \frac{1}{660n^{10}} + \dots \right) \quad (13) \end{aligned}$$

The coefficients of the denominators of the series are formed as follows:

$$\begin{aligned}\frac{1}{2} + \frac{1}{4} - \frac{2}{3} &= \frac{1}{12} \\ \frac{1}{4} + \frac{1}{6} - \frac{2}{5} &= \frac{1}{60} \\ \frac{1}{6} + \frac{1}{8} - \frac{2}{7} &= \frac{1}{168} \text{ etc.}\end{aligned}$$

The law of formation is evident, and any number of values can readily be calculated. The series is, however, *very convergent* for all values of n greater than 1. Thus when $n=2$, five terms are sufficient to give an accurate value of R_2 . These terms are as follows:

$$\log R_2 = \log 2 - (.0208333 + .0010417 + .0000930 + .0000108 + .0000015) \text{ whence } R_2 = 1.95653.$$

When n equals 3 or 4, four terms suffice; when n is 5 or 6, three terms suffice, and for larger values of n two or even one term is sufficient. Thus—

$$\begin{aligned}\log R_5 &= \log 5 - (.0033333 + .0000267 + .0000004) \\ \log R_6 &= \log 8 - (.0013021 + .0000041) \\ \log R_{11} &= \log 14 - (.0004252 + .0000004) \\ \log R_{24} &= \log 24 - (.0001447 + .0000000)\end{aligned}$$

The following values of the geometric mean distances (calling a unity) for the case under consideration were thus found:

$R_0 = 0.22313$	$R_5 = 4.98323$
$R_1 = 0.89252$	$R_6 = 5.98610$
$R_2 = 1.95653$	$R_7 = 6.98806$
$R_3 = 2.97171$	$R_8 = 7.98957$
$R_4 = 3.97890$	$R_9 = 8.99076$

5. CALCULATION OF THE MUTUAL INDUCTANCES FOR THE ASSUMED WINDING OF FLAT STRIP.

We can now calculate the last nine terms of formula (3), using formula (5), but putting the above values of R successively in place of the values of b , which in the case of round wires were 0.1, 0.2, 0.3 cm.

$$\text{Thus, } M_{11} = 4\pi a \left\{ \left(1 + \frac{3}{16} \frac{(.089252)^2}{625} \right) \log \frac{200}{.089252} - 2 - \frac{1}{16} \frac{(.089252)^2}{625} \right\}$$

$$= 4\pi a \times 5.71463 \quad (14)$$

$$\text{and } \frac{18M_{11}}{4\pi a} = 102.8633.$$

Proceeding in this way we find the values for the ten terms of equation (3). These values are given in column 2 of Table I. The values previously found for these terms for the winding of round wire are given in column 3. The differences between the corresponding terms are given in column 4, the sum of these differences being 6.0014, or 1.25 per cent, which represents the error of the current sheet formula when applied to this particular winding of round wire.

TABLE I.

	For Strip	For Round Wires	Differences
10 L_1	71.0090	67.6720	3.3370
18 M_{12}	102.8634	100.8166	2.0468
16 M_{13}	78.8771	78.5254	.3517
14 M_{14}	63.1671	63.0344	.1327
12 M_{15}	50.6421	50.5787	.0634
10 M_{16}	39.9525	39.9189	.0336
8 M_{17}	30.4965	30.4779	.0186
6 M_{18}	21.9449	21.9346	.0103
4 M_{19}	14.0950	14.0898	.0052
2 M_{110}	6.8120	6.8099	.0021
Total	479.8596	473.8582	6.0014

Thus, we have the value of $\frac{L}{4\pi a}$ for flat strips, by the two formulæ, as follows:

By the current sheet formula (1), 479.8595

By the summation formula (4), 479.8596

The difference between these results, amounting to less than one part in a million, is inappreciable. This discrepancy is of quite a

different order of magnitude from the difference found between the current sheet formula and the summation formula for the case of round wires amounting to over 1.25 per cent. It is thus seen that this method of deducing the self-inductance of a coil by the method of summation is a practicable one, and in the case of a coil wound with flat strip it leads to correct results. There is no reason why it should not be equally exact in the case of round wires covered by insulation. We may therefore be sure that the value 473.8582 for the coil of ten turns of round wires is an accurate value for that case.

6. COIL OF LARGE NUMBER OF TURNS.

When the coil has a large number of turns of wire, it becomes impracticable to use the summation formula, because of the large number of terms to calculate. Thus, in the inductance standard of the Bureau of Standards having 661 turns there would be 661 terms to calculate. But we may calculate the differences shown in Table I, between the self and mutual inductances for round wires and for flat strip, and apply their sum as a correction to the value found by the current sheet formula, and so obtain the desired value with a moderate amount of labor; for the differences become very small after the distance becomes appreciable. Thus, at 1 cm in the above case the difference for a single pair of wires was only 1 per cent of its value at 1 mm. In calculating these differences in the case of the large coil, we should of course stop as soon as the difference becomes inappreciable. The standard of inductance of the Bureau of Standards has three sections, which may be used singly or in combinations. Thus, there are six different cases. The number of turns and length of each coil is given in Table II, together with the value of the inductances as calculated by the current sheet formula.

The mean radius of the coils is 27.0862 cm. The wire is round and has a diameter of 0.0634 cm bare, 0.0694 cm covered. As already indicated, the above values of the inductances, calculated by formulæ which are correct for a winding which is equivalent to a current sheet, are too great for a winding of round wires, in which the thickness of the insulation is small. We have found Coffin's correction to be wrong, as it makes the corrected value larger instead of smaller than the value for a current sheet. We shall now

TABLE II.

Coil	No. of Turns	Length	Inductance by Current-Sheet Formulae
1	221	15.3347 cm	0.0361941 henry
2	251	17.3565 "	.0441703 "
3	189	13.1945 "	.0282220 "
1+2	472	32.6912 "	.112722 "
2+3	440	30.5510 "	.101810 "
1+2+3	661	45.8857 "	.179615 "

proceed to calculate the correction that must be applied to the above values to give the true values for this particular winding.

This correction consists of two parts and may be written as follows:

$$L_s - L = \Delta L_1 + \Delta M$$

where L_s is the value of the inductance calculated from the current sheet formula, L is the true value of the inductance, ΔL_1 is the correction depending on the self-inductance of the n single turns of the coil, and ΔM is the correction depending on the mutual inductances of each of the n turns on the $(n-1)$ other turns. The self-inductance of any coil of n turns may therefore evidently be written

$$L = L_s - \Delta L_1 - \Delta M$$

the corrections ΔL_1 and ΔM being subtracted from the value of the self-inductance given by the current sheet formula.

The total self-inductance of any coil of n turns may also be written

$$L = \Sigma L_1 + \Sigma M$$

where ΣL_1 is the sum of the self-inductances of all the n turns taken separately, and ΣM is the sum of the mutual inductances of each of the n turns on the $(n-1)$ other turns. The first term of the correction, ΔL_1 , is thus the difference between the value of ΣL_1 for a winding of flat strip which would exactly represent the current sheet and its value for the actual winding of round insulated wire; while the second term is the difference between ΣM for the winding of strip and the actual winding of wire.

7. TO CALCULATE THE CORRECTION ΔL .

Maxwell's formula for the self-inductance of a single circular turn of round wire, which is practically equivalent to Wien's, is derived from the formula for the mutual inductance of two parallel coaxial circles, by replacing b the distance apart of their planes by R the geometric mean distance of the section, which in this case is a circle. Thus

$$L = 4\pi a \left\{ \left(1 + \frac{3}{16} \frac{R^2}{a^2} \right) \log \frac{8a}{R} - 2 - \frac{1}{16} \frac{R^2}{a^2} \right\} \quad (17)$$

The geometric mean distance R for the circular section of straight wire is $\rho e^{-\frac{1}{4}} = .778801\rho$ where ρ is the radius of the section. This is not quite exact where the conductor is a circle, but is sufficiently exact when a is large and ρ relatively small. Wien's formula is derived directly by integrating the expression for the mutual inductance of two parallel circles (of infinitesimal section) twice over the area of the circular cross section of the conductor. By comparing the results of the two formulæ we may get an idea of the magnitude of the error arising from using the g. m. d. of the circle as the same as that for a rectilinear conductor. If $a = 25$ cm and $\rho = .05$ cm, L , the self-inductance of one turn of wire, is 654.40537π cm by Wien's formula and 654.40533π cm by Maxwell's. The difference is inappreciable. We need not hesitate, therefore, to use Maxwell's formula in the present case, where ρ is less than 0.04 cm.

The self-inductance L_1 of a round wire of radius ρ is given by formula (17), where R will be written R_1 and will have a value 0.778801 ρ . Similarly, the self-inductance L_2 of a single turn of flat strip of width d wound on a circle of radius a will be given by the same equation, except that R (here written R_2) will have a value 0.223130 d . The difference between the two will be

$$L_2 - L_1 = 4\pi a \log \frac{R_1}{R_2}, \quad (18)$$

neglecting small quantities, which here amount to less than one part in a million. Where the bare wire has a diameter 0.0634, $\rho = .0317$. The strip has a width D equal to the diameter of the covered wire, $= 0.0694$.

Hence

$$\begin{aligned} R_1 &= .778801 \rho \\ \text{and } R_2 &= .223130 D \\ \therefore \frac{R_1}{R_2} &= 1.594. \end{aligned}$$

Thus the excess of the self-inductance of the n turns taken separately of a coil wound with flat strip of width 0.0694 cm over the self-inductance of the same number of turns of round wire of 0.0634 cm diameter (0.0694 covered) is

$$\begin{aligned} \Delta L_1 &= 4\pi an \log_e \frac{R_1}{R_2} \\ \text{or, } \Delta L_1 &= 4\pi an \log_e 1.594. \end{aligned} \quad (19)$$

In the standard of inductance under consideration $a = 27.0862$ cm, and the number of turns of wire in each of the six sections is given in Table II. Substituting these values in equation (19), we have the following values of the corrections ΔL :

Coil 1	$\Delta L_1 = 35073$ cm
" 2	$\Delta L_2 = 39833$ "
" 3	$\Delta L_3 = 29993$ "
" 1+2	$\Delta L_4 = 74904$ "
" 2+3	$\Delta L_5 = 69827$ "
" 1+2+3	$\Delta L_6 = 104900$ "

8. TO CALCULATE THE CORRECTION ΔM .

The corrections ΔM are found in a similar manner. The mutual inductance of two parallel circles of round wire is given by equation (5), where b is the distance apart of the centers of the wires, which is also the geometric mean distance, supposing the radius a is large and b relatively small. In the case of the two strips, forming part of a current sheet, we have found the g. m. d. to be less than the distance apart of their centers. The mutual inductance of two such strips will therefore always be greater than that of two round wires, the distance apart of their centers being supposed the same in each case. The mutual inductance in the case of the wires will be,

$$M_1 = 4\pi a^2 \left\{ \left(1 + \frac{3}{16} \frac{b^2}{a^2} \right) \log \frac{8a}{b} - 2 - \frac{1}{16} \frac{b^2}{a^2} \right\} \quad (20)$$

and for the strip, putting kb for the geometric mean distance of the strips, where k is always less than unity,

$$M_2 = 4\pi a \left\{ \left(1 + \frac{3}{16} \cdot \frac{k^2 b^2}{a^2} \right) \log \frac{8a}{kb} - 2 - \frac{1}{16} \cdot \frac{k^2 b^2}{a^2} \right\} \quad (21)$$

The difference is

$$M_2 - M_1 = 4\pi a \left\{ \log \frac{1}{k} + \frac{1}{16} \cdot \frac{b^2}{a^2} \left(1 - k^2 + 3k^2 \log \frac{8a}{kb} - 3 \log \frac{8a}{b} \right) \right\} \quad (22)$$

In the above expression for $M_2 - M_1$, k does not differ from unity appreciably except where b is very small; that is, where the strips are very near together. Thus the second part of the expression is negligible in all cases, for when the coefficient $\frac{1}{16} \frac{b^2}{a^2}$ is more than 0.000001 (its value for $b=1$ mm) k is so nearly unity that the quantity within the parentheses is very small. Thus, for $b=1$ cm, the coefficient is 0.0001, $k=0.999575$ and the quantity in the parentheses is about 0.001, so that the term amounts to 0.0000001 and can be neglected. The correction δM for any pair of wires is thus

$$\delta M = 4\pi a \log \frac{1}{k} \quad (23)$$

Since the value of k depends upon the distance apart of the two turns of wire or strip under consideration, it is evident that there will be as many different terms as there are different distances. Thus

$$\begin{aligned} \delta M_1 &= 4\pi a \log \frac{1}{k_1} = 4\pi a \delta_1 \\ \delta M_2 &= 4\pi a \log \frac{1}{k_2} = 4\pi a \delta_2, \text{ etc.} \end{aligned} \quad (24)$$

where δM_1 is the correction for a pair of adjacent wires, the distance of their centers being D , the diameter of the covered wire, and Dk_1 their g. m. d.; δM_2 is the correction for a pair of wires distant $2D$, $2Dk_2$ being their g. m. d., etc. If there are n turns of wire on the coil we shall have

$$\begin{aligned}\Delta M &= 2(n-1)\delta M_1 + 2(n-2)\delta M_2 + 2(n-3)\delta M_3 + \dots \\ &= 8\pi a [(n-1)\delta_1 + (n-2)\delta_2 + (n-3)\delta_3 + \dots] \quad (25)\end{aligned}$$

In a coil of n turns there would be $(n-1)$ terms in this equation, but inasmuch as k rapidly approaches unity when the distance is increased, the correction terms decrease rapidly in value, so that only a limited number of terms need be calculated.

In Table III the consecutive values of the geometric mean distances are given up to R_9 and then every fifth value is given up to

TABLE III.

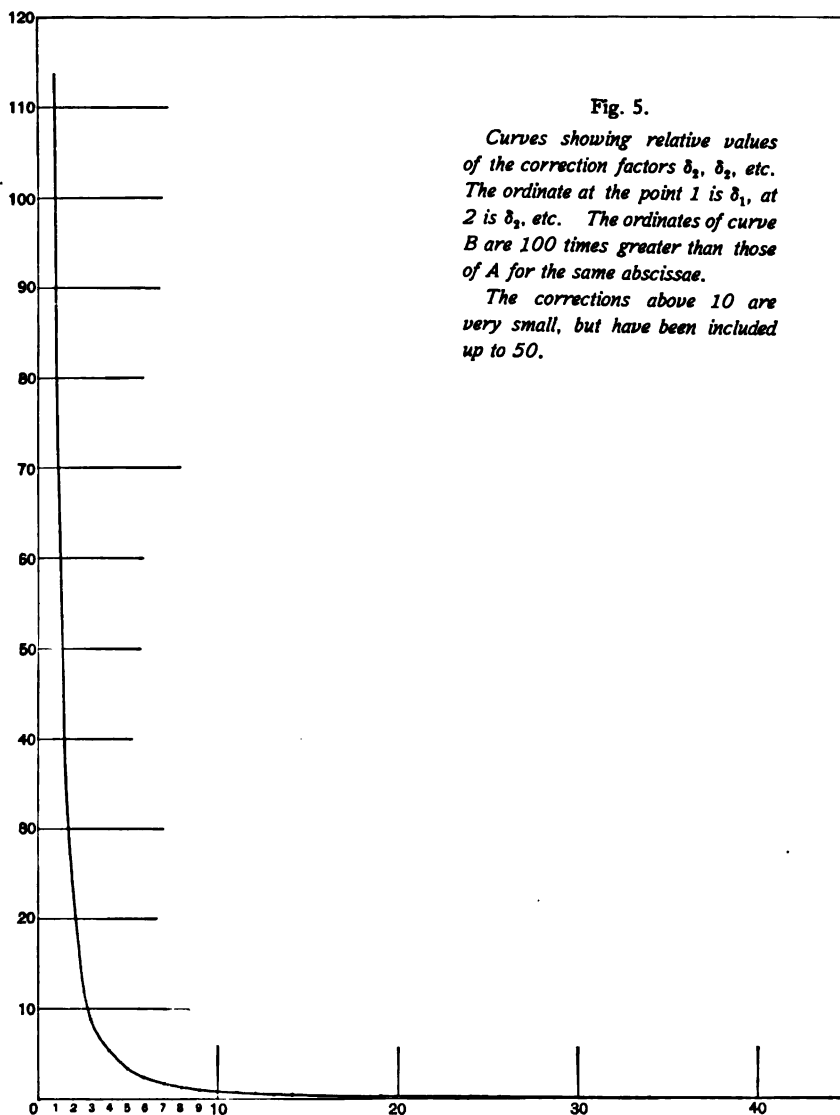
Geometric Mean Distances and Corrections Depending upon Them.

Geometric Mean Distances R	$k = \frac{R_n}{n}$	$\frac{1}{k}$	$\delta = \log_e \frac{1}{k}$
R_1 0.89252	0.89252	1.12042	0.11371
R_2 1.95653	.978265	1.02223	.02198
R_3 2.97171	.99057	1.00952	.00948
R_4 3.97890	.99473	1.00529	.00528
R_5 4.98323	.99665	1.00336	.00336
R_6 5.98610	.99768	1.00233	.00233
R_7 6.98806	.99829	1.00171	.00171
R_8 7.98957	.99869	1.00131	.00131
R_9 8.99076	.99897	1.00103	.00103
R_{14} 13.99405	.999575	1.000425	.000425
R_{19} 18.99531	.999753	1.000247	.000247
R_{24} 23.99653	.999855	1.000145	.000145
R_{29} 28.99724	.999905	1.000095	.000095
R_{34} 33.99754	.999928	1.000072	.000072
R_{39} 38.99785	.999945	1.000055	.000055
R_{44} 43.99811	.999957	1.000043	.000043
R_{49} 48.99828	.999965	1.000035	.000035
R_{99} 98.99920	.999992	1.000008	.000008

R_{99} , corresponding to the first and fiftieth turns of wire, respectively.

The second column gives the coefficients k obtained by dividing each geometric mean distance by the corresponding distance between centers of the wire, indicated by its subscript; thus, R_1 is divided

by 2, etc. The third column gives the values of $\frac{1}{k}$ and the fourth column gives the corrections δ , the natural logarithms of $\frac{1}{k}$. These



values are plotted in Fig. 5. The curve shows the values of δ up to $n=45$. This curve shows how rapidly the quantity δ falls off

as the distance increases. For the first and fiftieth wires it is only three ten-thousandths of its value for the first pair of wires, whereas the correction for the first and one-hundredth wire is less than one ten-thousandth of the first pair.

In Tables IV and V the products $(n-1)\delta_1$, $(n-2)\delta_2$, etc., are given

TABLE IV.

Corrections to Mutual Inductance.

B. S. Standard Coll. Sections 1, 2, 3.

$\text{Log } \frac{1}{k}$	Coil 1, $n=221$		Coil 2, $n=251$		Coil 3, $n=189$	
	$n-1$	Product	$n-1$	Product	$n-1$	Product
$\delta_1=.11371$	220	25.016	250	28.428	188	21.377
$\delta_2=.02198$	219	4.814	249	5.473	187	4.110
$\delta_3=.00948$	218	2.066	248	2.351	186	1.763
$\delta_4=.00528$	217	1.146	247	1.304	185	0.977
$\delta_5=.00336$	216	0.726	246	0.827	184	0.618
$\delta_6=.00233$	215	0.501	245	0.571	183	0.426
$\delta_7=.00171$	214	0.366	244	0.417	182	0.311
$\delta_8=.00131$	213	0.279	243	0.318	181	0.237
$\delta_9=.00103$	212	0.218	242	0.249	180	0.185
$\Sigma\delta_{10-14}=.00321$	209	0.671	239	0.767	177	0.568
$\Sigma\delta_{15-19}=.00163$	204	0.332	234	0.381	172	0.280
$\Sigma\delta_{20-24}=.00096$	199	0.191	229	0.220	167	0.143
$\Sigma\delta_{25-100}$	-----	0.493	-----	0.576	-----	0.405
	$\frac{\Delta M}{8\pi a} = 36.819$		41.882		31.400	
	$8\pi a=680.75, \therefore \Delta M = 25,064 \text{ cm}$		28,511 cm		21,375 cm	

for the six sections of the N. B. S. standard of inductance, as indicated in equation (25). The sum of the terms and $8\pi a$ times the sum are given for each coil, the latter being the correction ΔM for mutual inductance. Each term is given separately up to δ_9 , after that the sum of five consecutive terms is given, as $\Sigma\delta_{10-14}$, and finally the correction for all the turns from 25 to 100 is given in one. It will be noticed that the first four terms amount to about 90 per cent of the whole correction, and that the sum of the terms after δ_{20} amounts to only a little more than 1 per cent of the whole correction, or 1 part in 100,000 of the whole inductance.

The geometric mean distances given are those between parallel straight wires, and not parallel circles. But the quantity k is the ratio between the g. m. ds. of flat strips and round wires, and this

TABLE V.

Corrections to Mutual Inductance.

B. S. Standard Coil. Sections 1+2, 2+3, 1+2+3.

$\text{Log } \frac{1}{k}$	Coils 1+2, $n=472$		Coils 2+3, $n=440$		Coils 1+2+3, $n=661$	
	$\frac{n-1}{(n-2)}\text{etc.}$	Product	$\frac{n-1}{(n-2)}\text{etc.}$	Product	$\frac{n-1}{(n-2)}\text{etc.}$	Product
$\delta_1=.11371$	471	53.557	439	49.919	660	75.049
$\delta_2=.02198$	470	10.331	438	9.627	659	14.485
$\delta_3=.00948$	469	4.446	437	4.143	658	6.238
$\delta_4=.00528$	468	2.471	436	2.301	657	3.469
$\delta_5=.00336$	467	1.569	435	1.462	656	2.204
$\delta_6=.00233$	466	1.086	434	1.011	655	1.526
$\delta_7=.00171$	465	0.795	433	0.740	654	1.118
$\delta_8=.00131$	464	0.608	432	0.566	653	0.855
$\delta_9=.00103$	463	0.477	431	0.444	652	0.672
$\Sigma\delta_{10-14}=.00321$	460	1.477	428	1.374	649	2.083
$\Sigma\delta_{15-19}=.00163$	455	0.742	423	0.689	644	1.050
$\Sigma\delta_{20-24}=.00096$	450	0.432	418	0.401	639	0.614
$\Sigma\delta_{25-100}$	-----	1.180	-----	1.095	-----	1.700
	$\frac{\Delta M}{8\pi a} = 79.171$		73.770		111.063	
$8\pi a=680.75$, $\Delta M =$	53,896 cm		50,219 cm		75,606 cm	

can not be appreciably different for the case of parallel circles at moderate distances from its value for straight conductors. For the difference in the g. m. ds. is very small, and the difference in their ratios would be a small quantity of the second order. Hence the corrections calculated by (25) and given in Tables IV and V must be very exact.

In Table VI is given a summary of the values of the two corrections ΔL_1 and ΔM for the six sections of the standard coil, together with the values of the inductances given by the current sheet formula and the final corrected values. It will be noticed that the

whole correction varies in amount from 1 part in 600 for the smaller section to 1 part in 1,000 for the whole coil. This is a very large quantity in a standard when we remember the extreme

TABLE VI.

Summary of Corrections ΔL_1 and ΔM for the Six Sections of the N. B. S. Standard of Inductance, and the Corrected Values of the Inductances

	Coil No. 1 n=221	Coil No. 2 n=251	Coil No. 3 n=189
Correction for self-inductance ΔL_1	35073 cm	39833 cm	29993 cm
Correction for mutual inductance ΔM	25064 "	28511 "	21375 "
Total correction ΔL	60137 "	68344 "	51368 "
Total correction ΔL	0.0000601 henry	0.0000683 henry	0.0000514 henry
Inductance by current sheet formula L_s	0.0361941 "	0.0441703 "	0.0282220 "
Corrected inductance L	0.0361340 "	0.0441020 "	0.0281706 "

	Coils 1+2 n=472	Coils 2+3 n=440	Coils 1+2+3 n=661
Correction for self-inductance ΔL_1	74904 cm	69827 cm	104900 cm
Correction for mutual inductance ΔM	53896 "	50219 "	75606 "
Total correction ΔL	128800 "	120046 "	180506 "
Total correction ΔL	0.000129 henry	0.000120 henry	0.000180 henry
Inductance by current sheet formula L_s	0.112722 "	0.101810 "	0.179615 "
Corrected inductance L	0.112593 "	0.101690 "	0.179435 "

precision with which the measurements of the dimensions were made and the high sensibility obtainable in measuring self-inductance.

9. THE CORRECTION TABLES.

It is possible to put these two correction terms into such form that they may be quickly applied to any single layer coil, by the aid of tables of constants. Since $0.77880\rho = 0.3894d$, where d is the diameter of the bare wire, equation (19) may be written

$$\begin{aligned}\Delta L_1 &= 4\pi an \log_e \frac{0.3894d}{0.22313D} \\ &= 4\pi an \log_e \left(1.7452 \frac{d}{D} \right)\end{aligned}\quad (26)$$

From equation (25) we have

$$\Delta M = 8\pi a \sum_{n=1}^{n-50} \left(n \log \frac{1}{k} \right)$$

The sum of these two terms may be written

$$\Delta L = \Delta L_1 + \Delta M = 4\pi an [A + B] \quad (27)$$

where A stands for $\log_e \left(1.7452 \frac{d}{D} \right)$ and B stands for $\frac{2}{n} \sum \left(n \log \frac{1}{k} \right)$,

the summation being carried from $n = (n-1)$ to $n = 1$ for coils of less than 50 turns and up to $(n-50)$ for coils of more than 50 turns.

The values of the constants A are given in Table VII with the ratios $\frac{d}{D}$ as arguments. D is the width of the current sheet corresponding to one turn of wire. If the wire is wound so that the consecutive turns are in contact, D is also the diameter of the insulated wire. The mean length of the coil divided by the number of turns gives the value of D to be used. The mean diameter of the bare wire is d . The two corrections ΔL_1 and ΔM are to be subtracted from L . When the ratio $\frac{d}{D}$ is less than about 0.57, A is negative and hence ΔL_1 is negative, and it is added; ΔM is always positive.

The values of the constant B are given in Table VIII with n the total number of turns in the coil as argument. By means of these two tables and equation (27) it is easy to find the correction for any particular case, as the following illustrations will show.

TABLE VII.

Values of Correction Term A , Depending on the Ratio $\frac{d}{D}$ of the Diameters of Bare and Covered Wire on the Single Layer Coil.

$\frac{d}{D}$	A	Δ_1	$\frac{d}{D}$	A	Δ_1
1.00	0.5568	100	.70	0.2001	144
.99	.5468	101	.69	.1857	146
.98	.5367	103	.68	.1711	148
.97	.5264	104	.67	.1563	150
.96	.5160	105	.66	.1413	152
.95	.5055	106	.65	.1261	155
.94	.4949	107	.64	.1106	157
.93	.4842	108	.63	.0949	160
.92	.4734	109	.62	.0789	163
.91	.4625	110	.61	.0626	166
.90	.4515	112	.60	.0460	168
.89	.4403	113	.59	.0292	171
.88	.4290	114	.58	.0121	174
.87	.4176	116	.57	— .0053	177
.86	.4060	117	.56	— .0230	180
.85	.3943	118	.55	— .0410	184
.84	.3825	120	.54	— .0594	187
.83	.3705	121	.53	— .0781	190
.82	.3584	123	.52	— .0971	194
.81	.3461	124	.51	— .1165	198
.80	.3337	126	.50	— .1363	
.79	.3211	127	.50	— .1363	1053
.78	.3084	129	.45	— .2416	1178
.77	.2955	131	.40	— .3594	1335
.76	.2824	133	.35	— .4928	1542
.75	.2691	134	.30	— .6471	1823
.74	.2557	136	.25	— .8294	2232
.73	.2421	138	.20	— 1.0526	2877
.72	.2283	140	.15	— 1.3403	4054
.71	.2143	142	.10	— 1.7457	
.70	.2001				

TABLE VIII.

Values of the Correction Term B , Depending on the Number of Turns of Wire on the Single-Layer Coil.

Number of Turns	B	Number of Turns	B
1	0.0000	50	0.3186
2	.1137	60	.3216
3	.1663	70	.3239
4	.1973	80	.3257
5	.2180	90	.3270
6	.2329	100	.3280
7	.2443	125	.3298
8	.2532	150	.3311
9	.2604	175	.3321
10	.2664	200	.3328
15	.2857	300	.3343
20	.2974	400	.3351
25	.3042	500	.3356
30	.3083	600	.3359
35	.3119	700	.3361
40	.3148	800	.3363
45	.3169	900	.3364
50	.3186	1000	.3365

10. EXAMPLES ILLUSTRATING THE USE OF THE CORRECTION TABLES.

Example 1.—Coil of 10 turns, radius 25 cm, length 1 cm, diameter of insulated wire 0.1 cm = D , diameter of bare wire 0.08 cm = d ; thus $\frac{d}{D} = 0.8$.

From Table VII, $A = 0.3337$

“ “ VIII, $B = 0.2664$

$$A + B = 0.6001$$

$$n(A + B) = 6.001$$

$$\therefore \Delta L = 4\pi \times 6.001.$$

This value of ΔL is to be subtracted from L_s to obtain the true value L of the self-inductance.

This is the value found already (p. ¹⁷⁰), where the correction was determined by calculating L by the summation formula (2) and L_* by the current sheet formula (1).

Example 2.—Coil of 50 turns, radius $a = 20$ cm, length $b = 5$ cm, $D = 0.1$ cm, $d = 0.075$ cm; thus $\frac{d}{D} = 0.75$.

By formula (1)

$$\begin{aligned} L_* &= 4\pi a \times 2500 \left\{ \left(1 + \frac{25}{32 \times 400} \right) \log_e \frac{160}{5} + \frac{25}{128 \times 400} - .50 \right\} \\ &= 4\pi a \times 2500 \left\{ \left(1 + \frac{1}{512} \right) \log_e 32 + \frac{1}{2048} - .50 \right\} \end{aligned}$$

$$L_* = 4\pi a \times 2500 \times 2.972945 = 4\pi a \times 7432.36.$$

From Table VII, $A = 0.2691$

“ “ VIII, $B = 0.3186$

$$A + B = 0.5877$$

$$n(A + B) = 29.39$$

$$\therefore L = L_* - AL = 4\pi a (7432.36 - 29.39)$$

$$4\pi a = 251.3274$$

$$\therefore L = 1860570 \text{ cm}$$

$$= 1.86057 \text{ millihenrys.}$$

The correction to L_* here amounts to 0.4 per cent.

Example 3.—As an extreme case to test the method we may calculate the self-inductance of a single turn of wire. Let us take the particular case already calculated by Wien's and Maxwell's formulæ, (4) and (7), page ¹⁷³. The radius $a = 25$ cm, the diameter of the bare wire = 1 mm. We may now assume that the wire is covered and that the diameter D is 2 mm. Then $\frac{d}{D} = 0.5$. In using Rayleigh's current sheet formula we take the length of the equivalent current sheet as equal to D . We thus have

$$\begin{aligned} L_* &= 4\pi a \left\{ \left(1 + \frac{.04}{32 \times 625} \right) \log_e \frac{200}{0.2} + \frac{.04}{128 \times 625} - 0.5 \right\} \\ &= 4\pi a \left\{ \left(1 + \frac{1}{500,000} \right) 6.907755 + \frac{1}{2,000,000} - 0.5 \right\} \\ &= 4\pi a \times 6.40777. \end{aligned}$$

From Tables VII and VIII $A = -0.1363$ and $B = 0$. Thus, since $n = 1$, $\Delta L = 4\pi a \times (-0.1363)$, and being negative is added to L . Hence

$$L = 4\pi a (6.40777 + 0.1363) = 4\pi a \times 6.54407 \\ = 654.407\pi.$$

This is practically identical with the values given by the other formulæ (p. 172), the slight difference being due to the fact that the correction term A is carried only to four places of decimals.

If we had taken the bare wire of diameter 0.1 cm as equivalent to a current sheet 0.1 cm long in the above formulæ for L , we should have obtained a different value for L , but in that case $\frac{d}{D}$ would be unity and A would be $+0.5568$. The resulting value of L would, however, be the same as before.

Example 4.—Take the first section of the B. S. standard. Here $n = 221$, $d = .0634$, $D = .0694$. Hence $\frac{d}{D} = .9135$.

$$\begin{array}{rcl} \text{From Table VII } A & = & 0.4663 \\ \text{" " VIII } B & = & 0.3332 \\ A + B & = & 0.7995 \\ n(A + B) & = & 176.690 \end{array}$$

$$\begin{array}{rcl} 4\pi a & = & 340.375 \\ \therefore \Delta L = 4\pi a n(A + B) & = & 60,141 \text{ cm} \\ & = & .0000601 \text{ henry.} \end{array}$$

This is practically identical with the correction calculated directly for this section (Table VI).

Taking the whole coil, for which $n = 661$, we have

$$\begin{array}{rcl} A & = & 0.4663 \\ B & = & 0.3360 \\ A + B & = & 0.8023 \\ n(A + B) & = & 530.32 \\ 4\pi a & = & 340.375 \\ \Delta L = 4\pi a n(A + B) & = & 180,508 \text{ cm} \\ & = & .0001805 \text{ henry} \end{array}$$

which is the same value found previously for ΔL .

These examples are sufficient to illustrate the use and the accuracy of the correction Tables VII and VIII. By their use an accurate value of the self-inductance of a single layer coil of any number of turns can be calculated, if the proper current sheet formula is employed. Rayleigh's formula (1) already used is the most convenient one for short coils; that is, for coils whose length is small compared with the radius. Coffin's formula is an extension of Rayleigh's, and may be used where the length is too great to omit terms in $\frac{b^4}{a^4}$, $\frac{b^6}{a^6}$, and $\frac{b^8}{a^8}$ (b being the length and a the radius).

Lorenz's formula is an absolute one, and may be used for coils of any length, being more exact than Coffin's for coils whose length is as great as the diameter, but agreeing with Coffin's very exactly for all lengths up to $b=a$.

For convenience of reference I here give these three formulæ.

1. Rayleigh's formula:

$$L_s = 4\pi an^2 \left\{ \log \frac{8a}{b} - \frac{1}{2} + \frac{b^2}{32a^2} \left(\log \frac{8a}{b} + \frac{1}{4} \right) \right\} \quad (28)$$

2. Coffin's formula:

$$L_s = 4\pi an^2 \left\{ \log \frac{8a}{b} - \frac{1}{2} + \frac{b^2}{32a^2} \left(\log \frac{8a}{b} + \frac{1}{4} \right) - \frac{1}{1024} \frac{b^4}{a^4} \left(\log \frac{8a}{b} - \frac{2}{3} \right) \right. \\ \left. + \frac{10}{131072} \frac{b^6}{a^6} \left(\log \frac{8a}{b} - \frac{109}{120} \right) - \frac{35}{4,194,304} \frac{b^8}{a^8} \left(\log \frac{8a}{b} - \frac{431}{420} \right) \right\} \quad (29)$$

3. Lorenz's formula:

$$L_s = \frac{4\pi}{3} \frac{n^2}{b^3} \left\{ d(4a^3 - b^3)E + db^3F - 8a^3 \right\} \quad (30)$$

In the above formulæ a = radius of the coil and b = length of the coil, the length being the *mean over-all length including the insulation on the first and last wires*. In formula (3), d is the diagonal of the coil = $\sqrt{4a^2 + b^2}$ and E and F are the complete elliptical integrals of the first and second kinds, respectively, to modulus k , where $k = \frac{2a}{d} = \frac{2a}{\sqrt{4a^2 + b^2}}$. The subscript s is attached to L in each case as

a reminder that each is a current sheet formula, and the value of the inductance must in every case be corrected by formula (27) and Tables VII and VIII in order to give the true inductance of a winding of round wires. A winding of square or rectangular wire would also require correction, the only winding for which the formulæ are correct being a winding of strip of infinitesimal thickness in which the edges of the strip come together without making electrical contact, and so fulfilling the current sheet conditions assumed in deriving the formula.

In a subsequent paper I shall discuss the case of coils having more than one layer.

HEAT TREATMENT OF HIGH-TEMPERATURE MERCURIAL THERMOMETERS.

By Hobert C. Dickinson.

The ease with which mercurial thermometers can be used, their comparative simplicity and small cost, and the convenience of a direct reading temperature scale have led to the extension of their use to temperatures so high that the ordinary methods and precautions are insufficient. The apparent simplicity of the instrument often seems to hide the real sources of error.

Measurement of temperature, with greater or less accuracy, plays an important part in so many physical and chemical operations, both scientific and industrial, that not every user of a thermometer can be expected to be familiar with the peculiar properties of different kinds of glass and the various changes which may take place in them. If, however, it becomes necessary to determine with any approach to accuracy temperatures higher than about 300° C. (Centigrade degrees will be understood throughout this paper unless otherwise specified), special precautions must be taken, and a knowledge of some of the thermal properties of glass at high temperatures is desirable.

The precision attainable with mercurial thermometers at high temperatures is so largely determined by conditions of use that no definite limits can be assigned, but it may be said that to determine a temperature absolutely to within one degree at 450° requires considerable care. A thermometer may, however, repeat its readings more closely than one degree.

Great changes in indications may arise from continued use at high temperatures, even in thermometers which are properly made. These changes may generally be determined and proper corrections applied unless the glass is unsuitable or the instrument is not well made.

1. CONSTRUCTION OF HIGH-TEMPERATURE MERCURIAL THERMOMETERS.

Of the several kinds of glass commonly in use for thermometers, four are suitable for high-grade instruments. These are French hard glass ("verre dur"), Jena 16^{III} or "Normal" glass, Jena 59^{III} borosilicate thermometer glass, and Greiner and Friedrichs "Resistenzglas." Of these four the first two are not suited for use at temperatures much above 450° (840° F.). The Jena 16^{III} glass is more generally used in this country either for the whole thermometer or for the bulb alone with a stem of some other glass. Jena 16^{III} glass cracks easily from sudden changes of temperature or even with no apparent cause. This tendency, however, has not been observed in the bulbs. Above 450° it is necessary to use a harder glass. The one most used in this country seems to be the Jena 59^{III} borosilicate thermometer glass. The firm of Schott and Genossen, makers of the Jena glasses, has also developed other glasses (notably 122^{III}) suitable for high-range thermometers, but which have not yet come into general use. Thermometers are made from the 59^{III} borosilicate thermometer glass with scales running up to 550°, but experience has shown that they can not be used at this temperature continuously without undergoing great changes of the "ice point."¹ In fact, it is not safe to go above 530° for any great length of time.

When mercurial thermometers are used at high temperatures, some means must be provided to prevent distillation or boiling of the mercury. Even at 100°, if the top of the mercury column is heated and the tube above it is cool, particles of mercury collect slowly in the cooler parts of the stem. To prevent this below 200° it suffices to keep the top of the mercury column cool or the whole of the stem heated; but if the temperature is much in excess of 200° (390° F.), mercury will begin to boil in a thermometer sealed free from air if the top of the stem is cool. Boiling can be prevented at any temperature by filling the upper part of the stem with gas under sufficient pressure. The pressure required at 400° is about two atmospheres, at 550° twenty atmospheres, and at 750° sixty atmospheres. Two methods are in use for producing the necessary pres-

¹The term "ice point" is used to designate the reading on the thermometer immersed in melting ice, in place of the more ambiguous term "zero."

sure. These may for convenience be called the *small* upper bulb and the *large* upper bulb methods. The first is to make a very small bulb at the end of the tube or to leave a portion of the capillary at the end of the scale, and seal off at atmospheric pressure with the mercury standing at some lower point, say 0° . Evidently, if the size of the upper bulb is properly chosen the mercury on rising will compress the contained gas, increasing the pressure sufficiently to prevent boiling at all temperatures on the scale. It is not sufficient that the pressure be enough to prevent boiling at the highest points, but it must be determined for intermediate points as well. It might be too low at 450° and more than high enough at 550° .

The large upper bulb method has some marked advantages, though it is more difficult of application. In this method the end of the stem is provided with an auxiliary bulb or reservoir which is *large* in proportion to the volume of the stem itself, say twenty times the internal volume above zero degrees. This bulb is then filled with some suitable gas under sufficient pressure to prevent boiling of the mercury at the highest points on the scale. To seal off the top under pressure a little shellac or fusible metal is placed in a small tube connecting with the auxiliary bulb and is heated so that it flows down and seals up the small tube while still under pressure. The pressure may then be removed from the outside and the glass tube sealed off beyond the shellac in the ordinary way.

When the *small* upper bulb is used, the internal pressure increases rapidly as the temperature rises, depressing the readings of the thermometer. These variations will not have the same value in different instruments. When the *large* upper bulb is used, the pressure being nearly the same at all parts of the scale, the pressure correction is nearly a constant for any given instrument and does not affect the temperature scale defined by it. For filling the upper bulbs of high-temperature thermometers some inert gas, such as nitrogen or carbon dioxide, should be used free from moisture. If air is present, the mercury soon oxidizes and clings to the walls of the capillary. The use of pure mercury is also important.

The scale of temperature now in almost universal use is that of the hydrogen-gas thermometer, the degree Centigrade being defined as the one one-hundredth part of the change in pressure of a constant volume of hydrogen gas when its temperature is changed

between that of melting pure ice and that of steam from boiling pure water, the initial pressure of the gas at 0° being 100 cm of mercury, all under standard conditions. Above 100° the degree is similarly defined on the nitrogen-gas thermometer (see Bureau of Standards Circular No. 8). A mercurial thermometer may be graduated to define a temperature scale of its own, dependent only on the properties of mercury and the particular sample of glass. The conditions for such a thermometer are that the degree shall be represented by one one-hundredth of the volume of the stem contained between the readings at the melting point of ice and the steam point of water under standard conditions, and that the scale shall be extended in degrees of the same volume. Such a thermometer made from 59^{III} borosilicate glass defines a scale of temperature differing from the scale of the gas thermometer by amounts given below.²

TABLE I.

Variation from the Gas Scale for a Thermometer of Jena 59^{III} Glass.

<i>t</i>	<i>T</i>	<i>t</i>	<i>T</i>
0°	0	375°	385.4
100°	100	400°	412.3
200°	200.7	425°	440.7
300°	304.1	450°	469.1
325°	330.9	475°	498.0
350°	358.1	500°	527.8

where "*t*" represents temperature on the scale of the gas thermometer and "*T*" the corresponding reading on the glass thermometer as defined above. For Jena 16^{III} glass thermometers the variation from the gas scale is represented by Table II.³

Thus, if a thermometer is intended to read true gas scale temperatures, either the length of a degree must be changed at the higher points, as is generally done in working standards, or proper corrections must be applied. Toward the end of this paper a method is outlined for graduating a thermometer to read gas scale temperature.

²Mahlke: *Zs. für Instrumentenkunde*, 15, p. 178; 1895. Grützmaker: *Zs. für Instrumentenkunde*, 15, p. 250, 1895.

³Wiebe und Böttcher: *Zs. für Instrumentenkunde*, 10, p. 245; 1890.

TABLE II.

Variation from the Gas Scale for a Thermometer of Jena 16^{III} Glass.

t	T	t	T
0°	0	240°	240.46
100°	100.00	260°	260.83
150°	149.90	280°	281.33
200°	200.04	300°	301.96
220°	220.21		

2. CHANGES IN MERCURIAL THERMOMETERS.

If a thermometer be made with all the above precautions (i. e., proper kind of glass, pure mercury, and sufficient pressure of a neutral gas to prevent boiling) and so calibrated or corrected that it reads true gas scale temperatures at all points, there are a number of changes which will occur in the instrument due to lapse of time and conditions of use. These changes which take place, due to the physical properties of glass, are of two apparently distinct kinds: (1) After a thermometer is newly made there takes place a slow, permanent contraction of the bulb and stem which renders all the readings higher. This contraction is more rapid as the temperature is higher and may even raise the ice point several degrees per hour. (2) All thermometers show changes which are temporary in character. Thus, when a thermometer, after remaining for a long time at room temperature, is heated to 100°, the ice-point reading after this heating will be found *lower* than before, but this "depression" disappears in time if the thermometer is again kept at room temperature. In new thermometers the rise of the ice point may be sufficient to entirely hide this "depression." Thus, there seems to be a position of the ice point corresponding to each temperature. In thermometers of the better glasses, as those mentioned above, this position is reached in a minimum time. In general the time required to reach equilibrium when the temperature is *raised* is very small compared with the time required to reach equilibrium on *lowering* the temperature. For example, if a thermometer of "verre dur," or French hard glass, has been maintained at 0° until its ice point has reached its final position and is then heated to 100°, the position of

the steam point (or in other words the equilibrium condition of the glass) will be reached in ten minutes or less and the reading in ice after this heating will be about $0^{\circ}.1$ lower than before. But if now the thermometer be kept at 0° this depression will not entirely disappear for weeks. This is one reason why all thermometers should be made with the ice point included on the scale, so that these changes may be easily detected. For any of the glasses mentioned above these temporary changes of the ice point are comparatively small. *The depression for 100°* , i. e., the difference in the ice-point reading taken after a long time at 0° and immediately after 100° , is as follows:

TABLE III.

Value of Depression Constant.

"Verre dur" glass	$0^{\circ}.10$ to $0^{\circ}.07$
Jena 59 ^{III} glass	$0^{\circ}.02$
Jena 16 ^{III} glass	$0^{\circ}.07$ to $0^{\circ}.05$

Permanent changes of the first kind are, however, large in all glasses, reaching in some cases a total of 30° or 40° . They may affect the readings of a thermometer in four different ways: (1) The position of the fixed points of the scale, as the ice point and the steam point, may change. (2) The "fundamental interval" (the number of degrees of the scale comprised between the temperature of melting ice and that of steam from water boiling at normal pressure) may change. (3) The calibration corrections may change, i. e., the volume of the capillary stem per unit length may change by different amounts at different parts of the scale. (4) The temperature scale as defined by the mercury-in-glass thermometer may change relatively to the scale of the gas thermometer. The last two of these effects are so small that it is a question whether they can be determined in ordinary cases. Changes of the fixed points and the fundamental interval, on the other hand, are large, and it is necessary to take account of them.

Change in the fundamental interval may be due either to change in the volume of the capillary or change in the coefficient of expansion of the bulb. Apparently, both of these effects take place simultaneously, for, since the volume of the stem between 0° and 100° is

about one-sixtieth the volume of the bulb, if we assume that the same changes of volume take place in the stem as in the bulb the variation in the fundamental interval should be about one-sixtieth of the change in the position of the ice point. But, as will be seen from results which follow, the fundamental interval changes from two to three times that amount. For example, if the position of the ice point has changed from 0° to 3° , the steam point will probably have changed from 100° to $103^{\circ}.1$. Thus, the fundamental interval has changed from 100° to $100^{\circ}.1$, i. e., the change has been $1/30$ that of the ice point and not $1/60$, as might be expected.

The position of the ice point can change only on account of a change in the volume of the bulb or of the scale below the ice point. These changes are of far greater magnitude than any of the others, but fortunately they are the most easily determined and allowed for. One object of the present investigation is to determine the nature of the changes in various kinds of glass and to point out methods by which they can be rendered as small as possible.

It has long been known that the position of the ice point of mercurial thermometers may be rendered more permanent by continued heating at high temperatures, and some annealing or artificial aging process is often carried out in their manufacture. In 1837 Despretz⁴ made a statement of the general theory as follows: Whenever the molecules of a solid body suffer a displacement from a mechanical cause, as pressure, tension, or torsion, or from a physical cause, such as raising or lowering of temperature, they do not exactly return to their original state when the force is removed—that is to say, if the volume has been diminished or increased more or less by any force whatever it remains diminished or increased for a greater or less length of time after the force has ceased to act. Person⁵ was apparently the first to try annealing thermometers, both with and without internal pressure and he carried his temperatures higher than had been previously done. He found that the ice points rose as much as $17^{\circ}.2$ and suggested prolonged annealing as the remedy for all the defects of glass thermometers. His work,

⁴Guillaume. *Traité Pratique de la Thermométrie de Précision*, p. 138.

⁵Person *Comptes Rendus*, 19, p. 1314; 1844.

however, was not carried very far, and he supposed that long annealing would do away with the *temporary* as well as the permanent changes due to heating. Long annealing does not, however, eliminate the temporary changes. Kohlrausch⁶ showed that the recovery of the equilibrium condition in glass is much more rapid at higher temperatures. Crafts⁷ made a rather extended series of observations on eight thermometers, four of which were of French crystal and four of a lead-free soda glass. His observations extended over a period of about nine months and the conclusions were as follows: (1) Ice points of thermometers of French crystal rise more, and more rapidly, than those of lead-free soda glass. (2) Rise of the ice point is more rapid at first and probably tends toward constancy for a given temperature. (3) An ice point which is raised by long heating remains raised after cooling and the effect of the long heating is to render the ice point more stable for all lower temperatures. The numerical results are of little importance. In another paper this author records some experiments with a large glass balloon showing that at 511° internal pressure does not affect the annealing changes. He here makes application of the above theory of Desperetz to explain both the temporary depression and the permanent rise of the ice point thus, when a thermometer is made up before the blowpipe great strains arise in the glass. As the glass cools quickly from the very high temperature used in blowing, the residual strains are large and do not disappear for an indefinite time at ordinary temperatures, but if the temperature is raised these strains begin to be relieved with the result that the bulb contracts, *raising* the ice point. If now this heating be maintained long enough, all the previous strains may have disappeared, but if cooling takes place rapidly from this lower temperature of, say, 400° there will be residual strains corresponding to 400° which will disappear slowly at ordinary temperatures. If all these strains have had sufficient time to disappear, the glass will be free from strains or in equilibrium at ordinary temperatures. Upon heating to, say, 100° , and cooling quickly, it will retain strains corresponding to 100° and the position of the ice point will be *lower* after

⁶Kohlrausch: Pogg Annal. der Physik, 127, p. 4, 1866

⁷Crafts: Comptes Rendus, 91, pp. 291, 370, 413, 1880, 94, p. 1298, 1882.

heating to 100° than it was before. Thus both these apparently opposite variations in the position of the fixed points of a thermometer are referred to the same class of strains in the glass. Crafts also noted changes in the fundamental interval due to annealing.⁸

The work of these experimenters had solved some of the vexed questions in regard to the thermal properties of glass. (1) For every temperature there seemed to be a definite equilibrium condition, which a given sample of glass tends to assume more or less rapidly, depending upon the temperature, the previous history of the sample, and its chemical composition. (2) In general the equilibrium condition follows much more rapidly a rising than a falling temperature, but it takes place more quickly when the temperature is higher. (3) Long heating at higher temperatures renders the ice point more constant at lower temperatures. (4) The coefficient of expansion of glass is affected by its state of internal strains, being less by as much as 3 per cent for glass which is nearly free from strains than for glass which has been rapidly cooled. (5) Equilibrium conditions corresponding to small changes of temperature may occur without materially affecting the slow assumption of the equilibrium condition after more marked temperature changes.

With the above conclusions as a working basis a number of experiments have been carried out by different observers to gain a better knowledge of the peculiarities of different kinds of glass, to determine the magnitude of the changes in different samples and particularly to determine what kinds of glass best meet the requirements of the thermometer manufacturer; and a number of special thermometer glasses have been developed.

In 1883 Weber⁹ published the results of an investigation of the effect of chemical composition on the depression constant. He concluded that the chemical composition had a marked effect on the depression constant,¹⁰ that the softer kinds of glass were unsuitable for thermometers, and that the best results were obtained with a pure potassium glass containing large percentages of silicon and calcium. Wiebe¹¹ (1886) showed that the depression constant is greatest when

⁸ See also Pernet: *Comptes Rendus*, 91, p. 471; 1880.

⁹ *Ber. K. Akad. der Wissenschaften*, 18, p. 1233; 1883.

¹⁰ See Table III.

¹¹ *Zs. für Instrumentenkunde*, 6, p. 167; 1886, or "Jenaer Glas" by H. Hovestadt.

equal percentages of Na_2O and K_2O are present in the glass; that it is nearly the same, regardless of which is present in the greater quantity, provided the ratio of one to the other is numerically the same, and that it is least when K_2O only is present. The other constituents seem to have only a small effect on the depression constant.

Another series of experiments by Wiebe¹² on a number of thermometers of different kinds of glass led to the same general conclusions as those quoted above from Crafts. He also found that the glasses which show the smaller depression constant show the smaller rise of the ice point on heating. In a later paper he gives the results of some further experiments on five Jena 16^{III} glass thermometers. On heating to 500° these thermometers showed a marked permanent depression of the ice point due to enlargement of the bulbs under internal pressure, as the glass softens at that temperature.

Schott (1891)¹³ made an investigation to determine the temperatures at which different kinds of glass begin to soften. His observations were made on short pieces of rod ground plane at the ends and observed between crossed nicol prisms, the number of diffraction bands indicating the condition of strain. These samples had been first rapidly cooled, so that the strains were large. The temperature at which the Jena 16^{III} glass began to show unmistakable signs of softening was 400° and that at which the 59^{III} glass showed softening was 430°, although these glasses can be used for thermometers with from four to twenty atmospheres internal pressure at 450° and 530°, respectively, without showing any increase in the volume of the bulb. This author describes a method of rendering the strains in the glass visible. A rod of glass is surrounded by a liquid of the same index of refraction in a vessel with plate-glass sides. By examining this rod in polarized light the condition of strain is indicated by the number of diffraction bands which appear. A sample of Jena 59^{III} glass which showed innumerable bands before heating, after four days at 445° showed only four. The author shows also that slow cooling of a thermometer renders its subsequent changes smaller.

¹² Wiebe: *Zs. für Instrumentenkunde*, 8, p. 373; 1898; also 10, p. 207, 1890.

¹³ Schott: *Zs. für Instrumentenkunde*, 11, p. 330, 1891.

In Winkelmann's *Handbuch der Physik* (Vol. II., pp. 29-33, 62), there is given a résumé of some of the important work done on the thermal properties of glass, and "Jenaer Glas," by Hovestadt, gives a rather complete outline of most of the experimental work on the subject.

3. OUTLINE OF THE PRESENT INVESTIGATION.

Although considerable work has been done on the thermal properties of glass and valuable conclusions have been reached, most of the data obtainable are contained in scattered papers, few of them in English. The demand for information on this subject from a practical point of view as well as the hope of answering more fully some questions regarding the behavior of glass at high temperatures led to the present investigation. Some of the questions for which more definite answers are desirable are as follows:

- (a) How rapidly does the ice point rise at different temperatures?
- (b) How does this rate of rise change with time?
- (c) What is the effect of slow and of rapid cooling?
- (d) What is the effect of short periods of heating?
- (e) What process of heating will render the ice point sufficiently constant for different kinds of glass?
- (f) Can the annealing temperature be lower if the thermometer is to be used only at moderate heat?
- (g) What relation exists between changes in the fundamental interval and those of the ice point?
- (h) Does the use of a different kind of glass, having inferior thermal properties, for the stem of a thermometer introduce any significant errors?
- (i) Does the absence of pressure and mercury from the bulb of a thermometer during annealing affect its changes of volume?

The length of time covered by some of the following series of observations is far greater than in previous investigations.

4. APPARATUS.

Fifteen special thermometers (Fig. 1) were obtained in three different sets. The first set consists of four thermometers, three entirely of Jena 16^{III} glass and the fourth having a bulb of this glass joined to a stem of ordinary soft glass. The numbers of

these thermometers are as follows: G. 3689, 3690, 3691, and 3692. During the various operations all these except G. 3692, which has a soft glass stem, developed surface cracks and finally broke. The scales of all these thermometers consist of a portion of some 30° , 10° below and 20° above the 0° point, and the same at the 100° point separated by an auxiliary reservoir. Also, about the same length of scale is provided at the 200° point, with a continuous scale from 300° to 460° divided into single degrees. The second set of seven thermometers, Nos. G. 4279, 4280, 4281, 4282, 4290, 4291, and 4292, by the same maker, graduated in half degrees, consists of four having bulbs at the top large enough to contain all the mercury and having a scale covering about 30° above the 0° and above the 100° points, and three thermometers with scales running from 0° to 35° , from 100° to 135° , and from 400° to 470° . Two of the former and

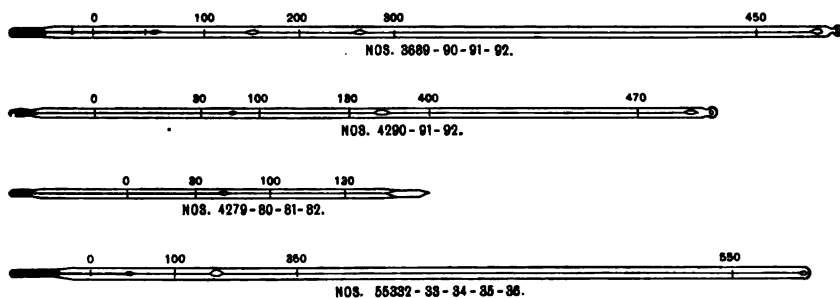


Fig. 1.—*Experimental Thermometers.*

one of the latter have soft glass stems. The other four are of Jena 16^{III} glass throughout. The third set consists of four thermometers of Jena 59^{III} glass, graduated in single degrees and having scales covering a few degrees at 0° , at 100° , and a continuous scale from 350° to 550° . In addition to the four thermometers of this set, the makers kindly loaned us two others of the same construction to be subjected to the same treatment. These six are numbered T. 55332, 55333, 55334, 55335, 55336, and 55337. In regard to all these instruments it was specifically required that the maker should not subject them to any process of aging, but where it was necessary to point the scale at higher temperatures some heating apparently was resorted to. The results of the experiments clearly show the effect of this heating.

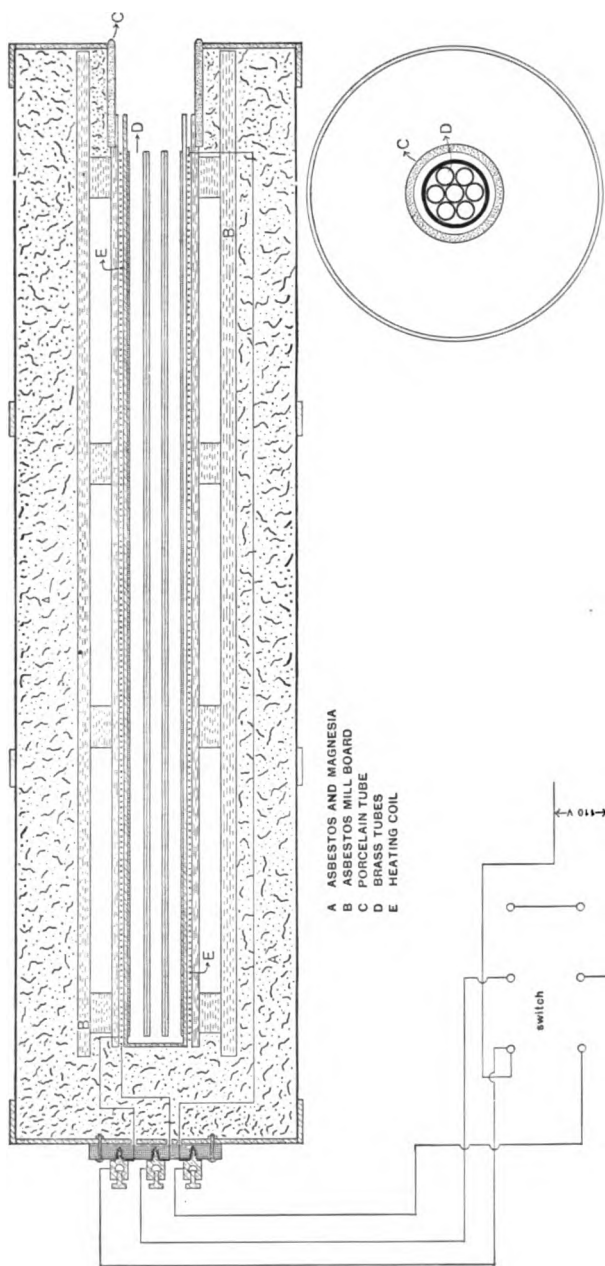


Fig. 2.—Annealing Oven, Showing Section and End View.

It was necessary to provide some means for maintaining high temperatures with fair constancy for long periods of time without undue attention. The annealing oven, which has been used for all the experiments of long duration and which is shown in Fig. 2, was built in the instrument shop of the Bureau of Standards and is of a simple form. A brass tube about 3 cm in diameter was selected to contain seven smaller tubes each large enough to hold a single thermometer. The larger tube was covered with a double layer of thin mica and wound with a *double spiral* of "Advance" resistance wire, No. 24 B. & S. The total number of turns of wire is 240, having a total resistance of about 60 ohms. With this arrangement three different heating combinations are possible. The two coils joined in series give a resistance of 60 ohms, either coil alone a resistance of 30 ohms, or the two in parallel a resistance of 15 ohms. It has been found that sufficient heat is produced with the coils connected in series directly to the lighting circuit of 120 volts to maintain the highest temperatures used, i. e., 550°. The other arrangements are made use of only when it is desired to raise the temperature rapidly. Care was taken to adjust the winding so that the temperature should be as nearly uniform as possible throughout the length of the oven. The brass tubes are about 60 cm long and are surrounded with a packing of magnesia and asbestos in a Russia iron cylinder about 16 cm in diameter and 75 cm long. With reasonable constancy of the lighting circuit the changes of temperature are only 10° or 15° at the most. About 100 watts will maintain 400° in this oven as it is arranged. A rheostat of 80 ohms resistance in steps of 10 ohms each is used in connection with a 10-ohm adjustable dial rheostat. In this way any desired temperature from 150° to 550° may be maintained. The accompanying cut (Fig. 2) shows a double-pole, double-throw switch connected up so as to throw the two portions of the heating coil in parallel or in series at will.

5. OBSERVATIONS.

The procedure in all the observations was to determine the readings of the thermometers in ice, then place those to be tested in the annealing oven, noting the time and temperature. After the desired length of time they were removed from the oven and the readings in ice taken as quickly as possible, except in cases where it was

desired to have cooling take place slowly. In these cases the oven was allowed to cool down by gradually diminishing the heating current until it reached nearly room temperature. Steam point readings were taken from time to time and the values of the fundamental interval computed. It should be noted that as the thermometers are divided only in degrees and half degrees the readings recorded in the tables are not to be depended upon nearer than about 0.03 or 0.04 . For this same reason the fundamental interval determinations are of only small value, since the changes which take place are often of the order of a few hundredths of a degree.

It could hardly be expected that a series of independent ice point and steam point determinations as long as the present one should not contain some errors. From the very nature of the tests no check could be had upon observations which had once been taken, and there appear a few points on the annealing curves which indicate that some error might have been made. In general, all observations were repeated two or three times at each point, but sometimes small bubbles of gas collect within the bulb of a thermometer which has been heated for a long time and these, if overlooked, introduce errors which no number of readings will reveal. In the case of the four thermometers made with large upper bulbs it was necessary to manipulate the mercury back into the lower bulb for each observation of the ice point, and it was difficult to tell when the last trace of gas had been worked out of the way.

The following table gives the observations taken on one set of the thermometers. In the column headed "Date" is given the time of the corresponding ice point or steam point determination. The column marked "Time" gives the number of hours or days during which the thermometers had been heated since the preceding observation. In the column headed "Remarks" are given briefly the conditions of the annealing, such as temperature of the oven, etc. For example, under series 1, the fifth line indicates that, after heating for 1.6 hour (or ten minutes) at a temperature of 200° the ice point of thermometer No. 3689 read -9.72 and the ice point of No. 3690 read -9.73 , observations being taken on October 24, 1904. Where boiling points are given they are reduced to standard conditions, i. e., what the readings would have been if the barometric pressure had been normal, making the temperature of the steam 100° .

TABLE IV.

Observations Taken on Four Jena 16^{ml} Glass Thermometers of Series I.

Date	368g	369g	369i	369a	Time	Remarks
1904						
Oct. 24	- 9.76	- 9.72	- 9.67	- 9.99		Zeros before heating.
24	90.03	89.97	90.03	89.78		Boiling point after 10 min.
24	- 9.79	- 9.75	- 9.71	- 10.02		Zeros after boiling point.
24	99.82	99.72	99.74	99.80		Fundamental intervals.
24	- 9.72	- 9.73			1/6 hr	At 200°.
24	- 9.66	- 9.65			1/6 hr	At 250°.
24	- 9.55	- 9.51			1/6 hr	At 300°.
24				- 10.00	2/3 hr	At 100°.
25	- 9.52	- 9.53			19 hrs	At room temperatures.
25	- 9.32				1/6 hr	At 325°.
25	Broke.	- 9.26			1/6 hr	At 350° (368g repaired).
25		- 5.03			1/6 hr	At 450° Hg. boiled (?).
25			- 9.72		1 hr	At 250°.
26			- 8.55	- 9.17	5 hrs	At 320°.
26			- 8.65		1/4 hr	At from 300° to 400°.
26		99.72				Fundamental interval.
26		- 5.73	- 8.57	- 9.14	20 hrs	At room temperature.
27		94.10	91.20	90.84	48 hrs	Boiling point after rest.
31		- 5.79	- 8.58	- 9.16	1/3 hr	After boiling point.
31		99.89	99.78	100.00		Fundamental interval.
31		- 3.27	- 5.72	- 6.58	22 hrs	At from 335° to 350°.
Nov. 1		96.69	94.32	93.58		Boiling point.
1		99.95	100.05	100.15		Fundamental interval.
1		- 3.26	- 5.73	- 6.57		After boiling for F. I.
2				- 4.68	24 hrs	At from 340° to 360°.
3				- 3.04	23 hrs	At from 350° to 360°.
3				97.18		Boiling point.
3				- 3.03		After boiling point.
3				100.21		Fundamental interval.
4				- 1.27	24 hrs	At 360°.
5				.48	24 hrs	At 360°.
7		- 3.26	- 5.67	- .47	48 hrs	At room temperature.
10				.07	72 hrs	At 340°.
14		- 3.23	- 5.69		14 days	At room temperature.
14				.38		Slowly cooled 36 h. Heated to 300°, 5 min.
15	- .93					First observation since repairing.
21		- 3.35		.27	1 hr	At 254°.
22		- 3.42		.18	1 hr	At 295°.
23	.77				22 1/2 hrs	At 297°.
25		- 3.35		.15	48 hrs	At room temperature.
25		- 3.28		.00	1 hr	At 340°.
26	25.1				23 hrs	At 445° and slowly cooled.
29		- 3.17		.11	1 hr	At 320°.
30		- 3.39			1 hr	At 367°.
Dec. 5		- 3.16			1 hr	At 375° on Dec. 1.
5			- 5.64		35 days	At room temperature.
1905						
Feb. 6	24.97	- 3.18	- 5.66	.18		At rest since last recorded observations.
15				8.46	21 hrs	At 450°.
17				11.23	21 hrs	At 450°.
Mar. 14				11.31	2 min	At 500°.
15				12.93	18 hrs	At 435°.
15				111.47		Boiling point.
15				12.93		After boiling point.
15				(96.54)?		Fundamental interval.
18				13.18		Continuous use up to 400°.
18		- 3.17	- 5.58			At rest since last recorded observations.
May 1				13.48		In general use.
18				13.26	1/2 hr	At 360°.
Aug. 22				13.38		In general use since May 24.

The accompanying curves (Figs. 3, 4, 5, 6, 7) give in graphic form the results of practically all the observations made on the ice points of the seventeen thermometers included in the experiments. The variations which take place during the first few hours are very much larger and more rapid than any which follow; unless the temperature be later increased. In order to show more clearly these early changes the first few hours are plotted on a much more open scale, so that Figs. 4 and 7 cover ten times the length of time of Fig. 3 and Fig. 6, respectively, and Fig. 5 covers five times the number of hours on Fig. 4 or fifty times the number on Fig. 3. For example, if the curves representing all the experiments in series No. 1 were plotted on the same scale as Fig. 3, their length would be about 25 meters. The ice-point readings are represented as ordinates and the number of hours annealing previous to the ice-point determination as abscissas. Since the position of the ice point is a complex function not only of the time, but also of the temperature, of annealing, and since it was desired to learn as much as possible of the nature of this complex function the temperatures were not in general maintained constant. The temperature of the annealing oven is therefore entered on each section of the curves, representing what should properly be a third dimension. For an example of the meaning of the curves it may be noted that the curve marked "G. 4281," beginning with Fig. 3 of series No. 1, should be interpreted as follows: Thermometer No. 4281 at the beginning of the experiments had a reading of $-0^{\circ}.1$ observed in melting ice; after heating for about 20 minutes at 100° (or in steam) the observed reading in ice was $-0^{\circ}.14$; then after 2 hours 20 minutes heating at a temperature of 380° the ice point was $4^{\circ}.58$; after 1 hour 40 minutes the ice point was $5^{\circ}.24$, and so on. In Fig. 6 of series No. 2 the curve marked "T.55336" indicates that during a heating of about 53 hours at a temperature of 528° the position of the ice point had fallen from $9^{\circ}.0$ to $-34^{\circ}.0$. This marked lowering of the ice-point reading is due to actual enlargement of the bulb of the thermometer from internal pressure and softening of the glass; hence, this temperature is too high to measure with this instrument. Series No. 1 gives the results of experiments on thermometers of Jena 16^{III} glass and series No. 2 on those of Jena 59^{III} glass.

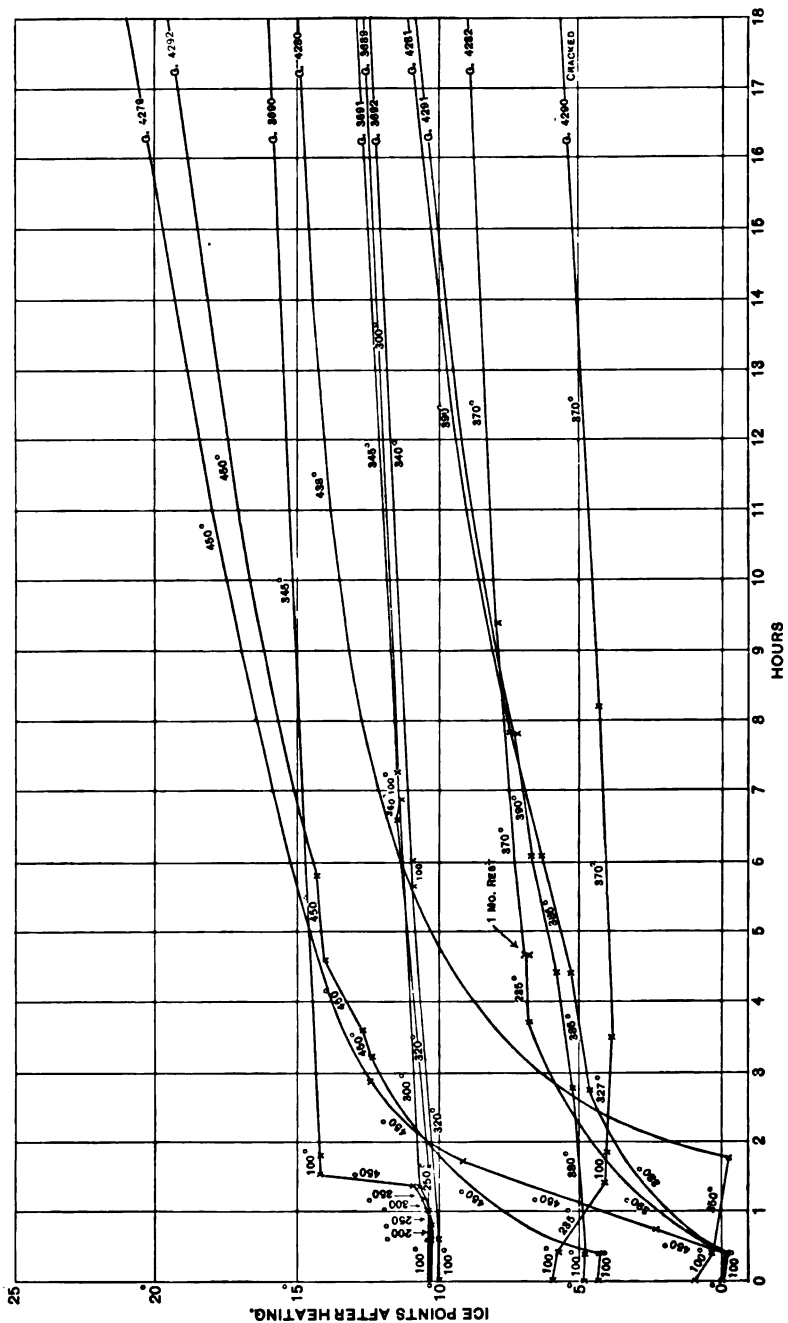


Fig. 3.—(Series 1.)

*Ice-point changes due to continued heating up to 18 hours at temperatures indicated on the curves.
Thermometers of Jena 1611, "normal" thermomater glass.*

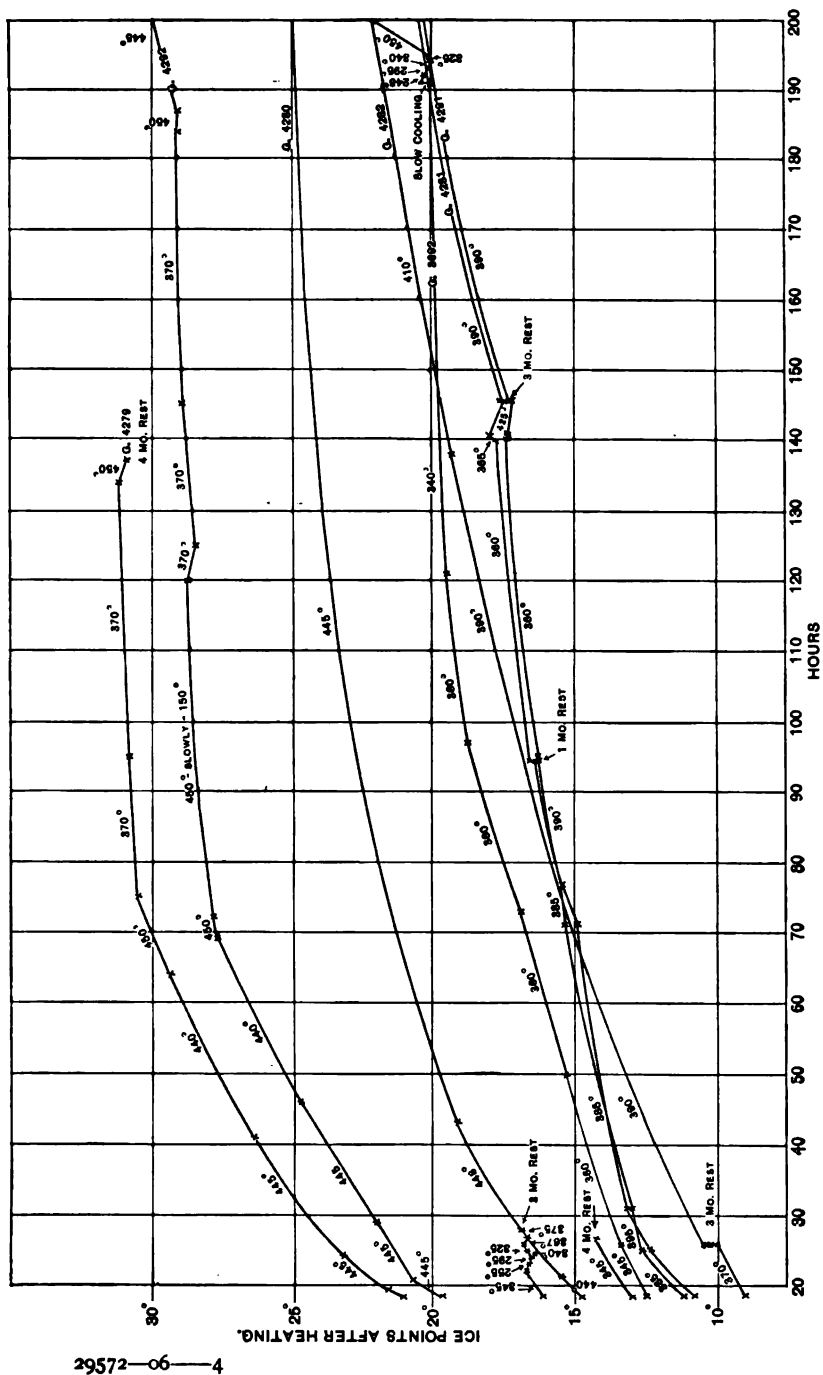


Fig. 4.—(Series 1—Continued.)
Ice-point changes due to continued heating up to 200 hours at temperatures indicated on the curves.
Thermometers of Jena 16^{III}, "normal" thermometer glass.

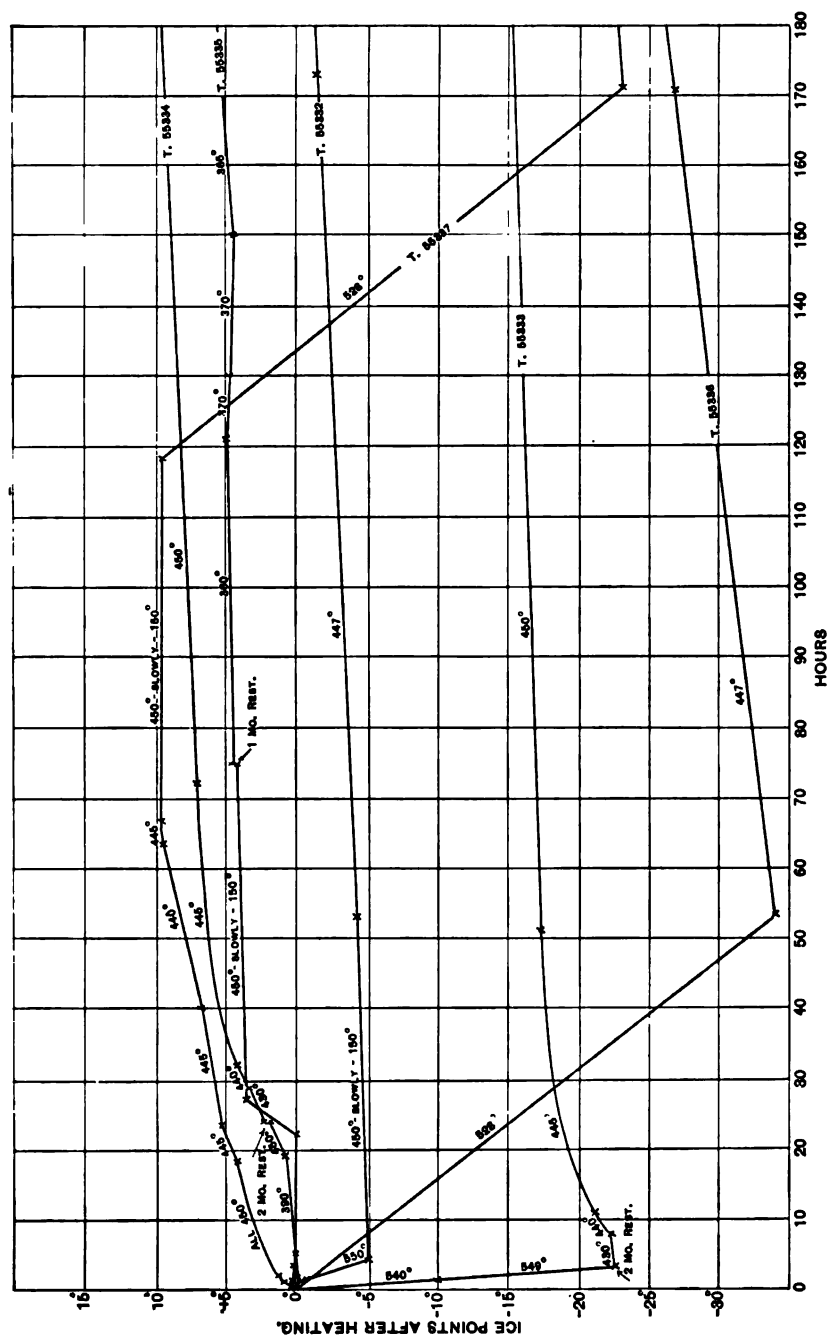


Fig. 6.—(Series 2.)
Ice-point changes due to continued heating up to 180 hours at temperatures indicated on the curves.
Thermometers of Jena 59^{III}, borosilicate thermometer glass.

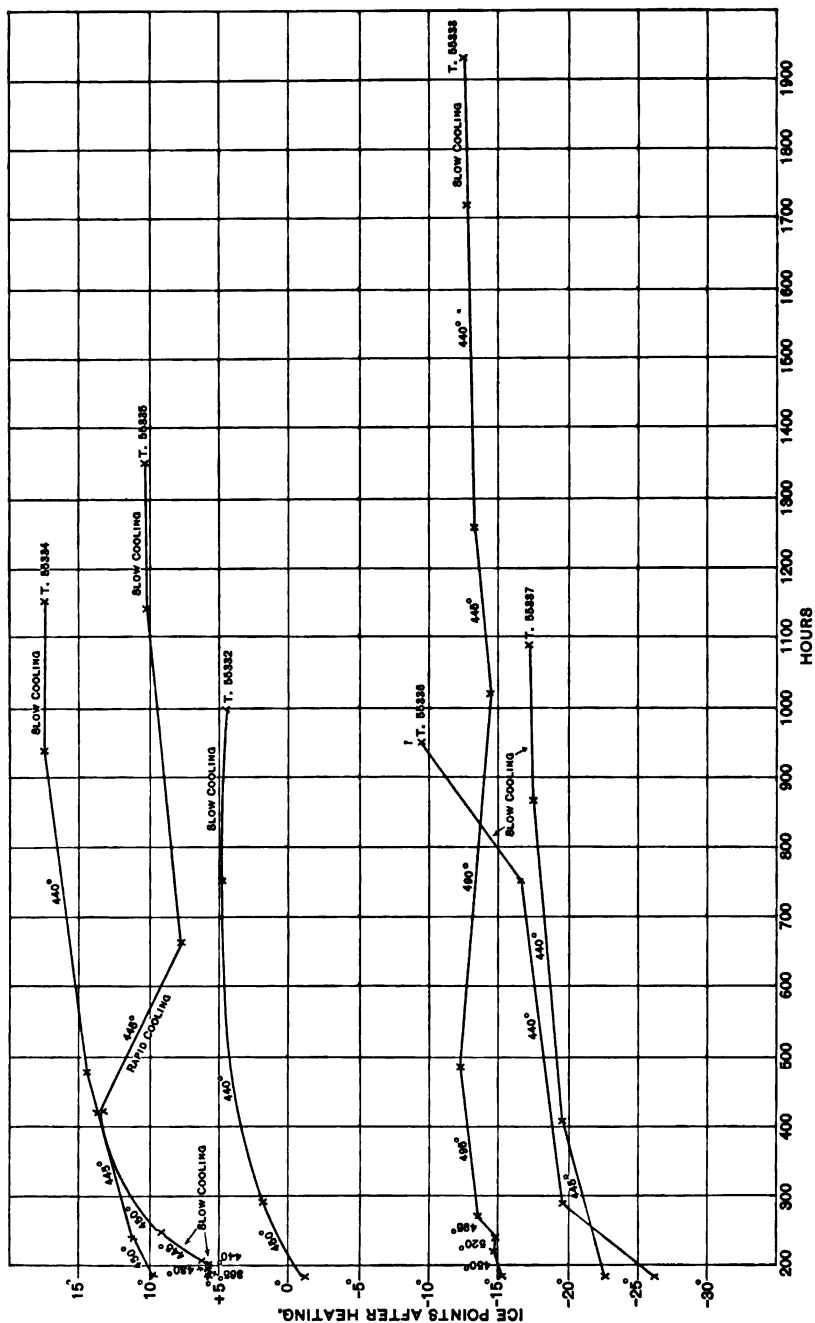


Fig. 7.—(Series 2—Concluded.)
Ice-point changes due to continued heating up to 2,000 hours at temperatures indicated on the curves.
Thermometers of Jena 59^m, borosilicate thermometer glass.

6. DISCUSSION OF RESULTS.

The results of the present experiments have been put in graphic form with the belief that in that way they could be most readily studied. The conclusions stated hereafter are founded on the results of previous investigations as well as those embodied in accompanying tables and curves, the examination of which leads to answers to some of the questions previously suggested.

(a) A newly filled thermometer which has not been subjected to heat, when first subjected to temperatures higher than that of the room will show a change of its ice point which may be in either direction and of varying magnitudes, depending upon the temperature, the nature of the glass, the length of time heated, and the previous treatment. If the temperature be not more than 100° or 150° and the heating continue only for some minutes the ice point reading will be lowered by a small amount (e. g., $0^{\circ}.1$ or $0^{\circ}.2$ for Jena "normal" glass). If the temperature be higher than about 150° for Jena "normal" glass or 400° for the borosilicate glass, or in any case if the time be long enough, the ice point reading will *rise*. This rise will be much more rapid at higher temperatures, the changes during the first hour being roughly as follows:

TABLE V.

Rise of the Ice Point during First Hour's Heating at Different Temperatures.

Jena 16 ^{III} Glass		Jena 59 ^{III} Glass	
Temperature	Ice Point Rise	Temperature	Ice Point Rise
200°	0°1	300°	-0°1 (drop)
300°	0°3	400°	-0°1 (drop)
350°	0°7	450°	+0°7
400°	2°5		
450°	8°0	550° shows a decided drop.	

Thus the change of ice point during the first hour's heating of a Jena 16^{III} glass thermometer at 450° is of the order of ten times that for the same time at 350° . About the same proportion would probably apply for 59^{III} glass between 425° and 525° . In the present series of experiments all the thermometers of this glass

showed a decided lowering of the ice point when heated much above 500° . Results obtained from the instruments after this lowering had taken place, due to internal pressure and softening of the glass, would obviously be uncertain.

The observations indicate that a certain amount of annealing took place at the same time that the bulbs were being enlarged by overheating, since the subsequent heatings changed the ice point less than if the thermometers had not been overheated. This effect may be seen by comparing the curves for T. 55333 and T. 55334 in series 2. The former had its ice point depressed by $22^{\circ}.5$ and in the course of 500 hours heating afterwards the ice point rose by about 10° , while the latter, which did not suffer such a lowering, shows a rise of about 15° in the same length of time and under less severe conditions of heating.

(b) The change of ice point is much more rapid during the first few hours, and becomes slower as time goes on; but apparently the time required to reach a limiting position is very long. Some of the thermometers have been observed after heating for a total of over 1,900 hours (80 days), and still show slow changes. The rates at which ice points change are shown approximately by the following table:

TABLE VI.

Rise per Hour of the Ice Point after Different Lengths of Time at Various Temperatures.

Jena 16 ^{III} Glass			Jena 59 ^{III} Glass		
Temperature	Time	Rise per Hour	Temperature	Time	Rise per Hour
350°	50 hours	0°1	400°	50 hours	0°02
400°	50 "	.07	450°	50 "	.02
450°	50 "	.05	500°	50 "	(.02)
350°	200 "	.01	400°	200 "	No data
400°	200 "	.006	450°	200 "	.01
450°	200 "	.003	500°	200 "	.008
			500°	500 "	.00 ?

This table shows that the change of the ice point is very slow after 200 hours' annealing, and for 59^{III} glass is almost inappreciable after 500 hours' heating at 500° .

(c) For temperatures a few degrees lower than that at which the annealing has taken place the changes are very small even after one hundred hours, but if the thermometer has been quickly cooled it will show a sudden small rise of the ice point on heating up to a temperature a little lower than that from which it was cooled. This effect is small but has been repeatedly observed even after a long time has elapsed. These observations after long rest indicate that the strains remaining after rapid cooling from, say, 400° or 500° do not disappear to any large extent at room temperature. For this reason it is best that a thermometer should be slowly cooled after annealing. If cooled slowly the glass has time to take up its equilibrium condition at each stage of the temperature drop. For example, when a sample of glass has been kept for a long time at 450° it will be nearly free from strains at this temperature, but on cooling strains are introduced. If quickly cooled to room temperature these strains will persist for a very long time, perhaps for years, but the strains left from heating at 450° will nearly all disappear at 430° in a few hours, and again the strains left from heating to 430° will soon disappear at 400° , and so on. At 25° the strains due to cooling from a temperature of 100° will be inappreciable after a few days. Thus it is possible to have the glass of a thermometer almost entirely free from strains at a temperature of 25° by this process, whereas if cooled rapidly this condition would not be reached in a very long time, perhaps not in years.

(d) If a number of similar unannealed thermometers originally reading alike at the ice point be subjected to different short periods of heating the ice-point readings will be very different. And if now these thermometers be put together in an annealing oven at a temperature higher than that to which they have been previously subjected, a few hours will suffice to make them all read nearly alike. Hence it seems that short periods of annealing at lower temperatures have no marked effect on the total time subsequently required to satisfactorily anneal a thermometer at a higher temperature; e. g., it would take as long at 400° to bring the ice-point reading up to $+25^{\circ}$, whether starting with a thermometer which had previously had its ice point raised to 15° or with one not previously heated, provided both read 0° in ice when newly made. Thermometers G. 4279 and G. 4292 or G. 4281 and G. 4291 (see Fig. 3), though they differed

by more than 4° at the start, came to the same ice-point reading after the first few hours' heating. The difference in the initial positions of the ice points is presumably due to the treatment by the maker in pointing G. 4291 and G. 4292 at the higher parts of the scale; G. 4279 and G. 4281 not being pointed above 100° were not thus affected. The same effect may be seen by comparing G. 3692 with G. 4279 and G. 4281. Though the first was annealed for many hours at a lower temperature, the time required to reach a given rate of rise at 445° was almost identical in all three cases. The previous history and original ice point of G. 3692 are not known, but it had probably not been heated above 445° . It seems to make little difference in the final results whether annealing takes place with continuous heating or in short periods of the same total length.

(e) As previously noted Schott has shown that total disappearance of the residual strains is possible at from 400° to 410° for Jena 16^{III} glass and 430° to 440° for Jena 59^{III} glass. The results of the present investigation seem in general to confirm this conclusion. Thermometers G. 4281, G. 4282, and G. 4291 after long annealing at about 410° seem to have reached the same condition as G. 4279 and G. 4292, which have been annealed at about 450° . From the observations on thermometers annealed at from 300° to 350° it appears that the time required at these temperatures to reach complete annealing or freedom from strains, which apparently would mean a rise of the ice point of about 30° , is excessively long if such a condition would ever be attained. There are no conclusive examples in the case of the Jena 59^{III} glass. In this case if the thermometers are to be used up to 500° it would probably be better to make the annealing temperature higher to save time, though a long enough period at 450° might accomplish the same result. In respect to length of time required, the present observations do not appear to agree with Schott's conclusions reached by his optical method. The time required to attain anything like constancy of the ice point has been found much longer than the four days which he found sufficient to relieve practically all the strains he could observe. It is possible that the residual strains which he observed, and which are referred to in the earlier part of this paper, were of greater importance than appeared, but it is not clear that one is justified in assuming that all the strains which give rise to changes

in the volume of glass as indicated by changes of the ice point are of such a nature that they can affect its optical properties as observed with polarized light.

(f) When a thermometer has been annealed for a short time at some high temperature, as, say, 450° , and is then kept at some lower temperature, the rate of rise of the ice point is much slower on account of this heating. Thus, a relatively short annealing renders the ice point much more stable for lower temperatures. This fact may be explained thus: For example, the total rise of the ice point during fifty hours' heating at 400° is 15° and the rate of rise after such heating is 0.05° per hour. Then if the thermometer were annealed at 450° (instead of 400°) for about seven hours the total rise of the ice point would be about the same (15°) and the rate of rise at 400° after this treatment would probably be about the same as if the annealing had been done at 400° , and had continued for fifty hours. In other words, the rate of change per hour at a given temperature, as 400° , will be nearly the same for thermometers of a given kind of glass when the ice point has been raised to any given extent, as, say, 15° , independently of whether that rise of the ice point has been produced by heating for a *short* time at a *high* temperature or for a *longer* time at a *lower* temperature. Thus if a thermometer is not to be used at the highest temperatures the time of annealing may be diminished or the temperature of annealing may be lower.

(g) The steam point and fundamental interval have been determined several times for each thermometer, but owing to the fact that the instruments are only divided to degrees or half degrees these determinations can not be relied upon to much better than 0.1° . It has been observed that there is an increase of the fundamental interval accompanying each rise of the ice point. The ratio of these two changes determined from the mean of all the observations taken on the 11 Jena glass thermometers is 0.035, or the fundamental interval increases by thirty-five thousandths (about one-thirtieth) of the change in the position of the ice point. This conclusion is in accordance with previous observations. The variation of the fundamental interval is of importance in the case of thermometers which show large changes of the ice point. Take, for example, a thermometer which shows a change of the ice point amounting to, say,

6°. The change in the fundamental interval will be about 0°·2. Since this is the additional correction to be applied at 100° and must be added again for each 100°, the correction at 450° will amount to 0°·9 (if the ice point reading is +6°, the reading at 450° will be 456°·9), a quantity which can by no means be neglected. It is therefore necessary to determine this correction whenever the ice point has shown a serious change. The method of slow cooling described above renders both the ice point and particularly the fundamental interval more stable, as previously mentioned.

(h) The use of some softer glass for the stems of thermometers in connection with bulbs of Jena 16^{III} glass is common in this country. To determine whether this practice introduces any important uncertainties in the use of such thermometers some of the experimental instruments were made up with stems of softer glass, as previously described. As changes of the ice point are due only to changes which take place in the volume of the bulb this practice could affect only the variations in fundamental interval or in calibration corrections. The latter changes are too small to be determined, and so far as can be observed with the experimental thermometers divided only to half degrees there is no difference between the changes of fundamental interval in the two cases. There is, however, a decided disadvantage in the use of the softer glass at very high temperatures. Above about 420° the glass of the stem becomes soft enough to bend. Whether this bending is due to the weight of the stem or to unequal contraction has not been definitely determined. Below 420° stems of the softer glass may be used with the advantage of being less liable to crack from sudden changes of temperature.

(i) The effect of internal pressure on the rise of the ice point is shown by comparison of curves for (Series I) thermometers G. 4279 and G. 4292, or the curves for G. 4281 and G. 4291; 4279 and 4281 being filled free from pressure and having large bulbs at the top to contain the mercury during the process of annealing. These results indicate that if the annealing temperature is lower than about 390° no difference is noticeable between the thermometers filled under pressure and those free from air. If the annealing temperature be raised to 410° or higher, the equality no longer holds true, but the thermometers filled under pressure read consistently lower. This may be due to the fact that the glass is actually softened to such an

extent that the internal pressure can have a slight effect in enlarging the bulb. According to Schott, Jena 16^{III} glass begins to show softening at about this point.

The results here obtained would indicate that in the case of Jena 16^{III} glass there should be a small depression of the ice point if a thermometer previously annealed free from pressure should be filled under pressure and further annealed. No data are at hand to show whether a depression actually would take place. At all temperatures between 400° and 450° this glass is apparently in a condition which is plastic as regards forces such as the internal stresses of the glass, but which is in no sense plastic as regards forces such as internal pressure. The internal strains practically disappear at these temperatures, while internal pressure produces no continuous effect which indicates plasticity, as it does, for instance, in the case of the Jena 59^{III} glass thermometers, which showed a continuous increase in the volume of the bulb as long as they were kept at a temperature of 520° or more. Whatever the effect of internal pressure may be in thermometers of Jena 16^{III} glass below 450° it is not large, and the observations indicate that it is possible to get good results by annealing a thermometer completely before it is filled with mercury. The subsequent changes will not cause serious trouble.

Two of the experimental thermometers were opened after the tests in such a way as to determine their internal pressure at different points of the scale. Table VII gives the results.

From this table it appears that the pressure in G. 4290 was insufficient to prevent boiling of the mercury at 450° as had been found. The vapor pressure of mercury is a trifle over 2 atmospheres at 400° and a trifle under 4.5 atmospheres at 450°.

In addition to the data given above, some results have been obtained through the kindness of Mr. L. Baudin of Paris, and the Taylor Brothers Company of Rochester, N. Y. In two letters Mr. Baudin reports the change of ice point which took place during his annealing of a total of twelve thermometers made for the Bureau of Standards from French hard glass (*Verre dur*). These thermometers are in four sets and the annealing temperature was about 445° or the temperature of sulphur vapor. The first four were heated for 72 hours showing a rise of the ice point of from 20°0 to 20°7, the next two were heated for 144 hours with a rise of 22°2 and 22°7,

TABLE VII.

Internal Pressure in Two of the Experimental Thermometers.

Thermometer G. 4390		Thermometer T. 55337	
Point on Scale	Pressure in Atmospheres	Point on Scale	Pressure in Atmospheres
200°C	1.1	200°C	4.2
300°	1.3	300°	5.4
400°	1.6	400°	7.7
470°	1.9	500°	13
		550°	21

the next four for 72 hours with a rise of from 18°1 to 19°5 and the last two for 144 hours with a rise of 21°5 and 22°0. These results show a considerably smaller annealing effect than found for Jena 16^{III} glass in the present series of experiments. The Taylor Brothers Company report that six 59^{III} glass thermometers similar to the ones used in the present tests showed a rise of ice point amounting to 59° F. (or 33° C) after heating for a total of 72 hours in 6-hour periods at a temperature of about 950° F. (510° C).

Owing to the fact that the Jena 59^{III} glass thermometers showed an enlargement of the bulb at temperatures somewhat lower than was expected, a careful analysis of the glass from one of them was made by Mr. J. R. Cain, of the chemical division of the Bureau of Standards, with the following results. For the sake of comparison the values found by Schott & Genossen for 59^{III} glass are given in another column.

The presence of a smaller percentage of B₂O₃ in the present sample may account for its lower softening temperature, and the presence of both Na₂O and K₂O would indicate that the depression constant is larger in the present case, but this could not be determined with thermometers graduated only in single degrees. The American manufacturers of these thermometers have given us their assurance that the glass used was sent to them direct from the firm of Schott & Genossen as borosilicate glass. Thus the above analysis is probably representative of the product which can be obtained by American makers.

TABLE VIII.

Chemical Analysis of Samples of Jena 59^{III} Glass.

	B. S. (Cain)	S. & G.
SiO ₂	72.30 per cent	71.95 per cent
Al ₂ O ₃	6.14 " "	5.00 " "
Na ₂ O	9.51 " "	11.00 " "
K ₂ O	1.32 " "	
B ₂ O ₃	10.10 " "	12.00 " "
Mn ₂ O ₃	0.03 " "	.05 " "
	99.40	100.00 " "

7. Important Precautions.

In the foregoing pages, questions suggested have been discussed and conclusions have been drawn from the results of the present investigation. It is, however, deemed worth while to summarize briefly here the points of more immediate and practical importance.

(1) *Jena 59^{III} borosilicate is the best thermometric glass in use, particularly for high temperatures, but it can not safely be used at much above 500°, when its composition is that of the above sample.*

(2) *Jena 16^{III} glass can be used up to 450° or somewhat higher with good results and for temperatures lower than 420° a softer glass may be used for the thermometer stem.*

(3) *Every thermometer, particularly if intended for use above 100°, should undergo a suitable system of annealing or artificial aging before it is used. The annealing can be done before the thermometer is filled. Thorough annealing requires from four to ten days at about 450°, according to the temperature at which the thermometer is to be used, and the annealing may well be followed by a period of slow cooling for from three to six days. (The latter is, however, less important.) The total change of the ice point for this process of annealing will be about 30° for Jena 16^{III} glass and from 20° to 30° for Jena 59^{III} glass.*

(4) *The ice point of a thermometer will change with use at higher temperatures and must be determined occasionally if accurate results are to be had. If the thermometer has not been annealed at tem-*

peratures above 400° for 16^{mm} glass or 430° for 59^{mm}, the changes will be large if it is heated above the temperature at which it has been annealed. If changes have taken place in the ice-point reading, the fundamental interval has probably changed by about 3 per cent of this amount and the resulting error at 500° may be as much as one-fifth of the change of the ice point. [See (g) section 6.]

(5) To prevent boiling of the mercury in a thermometer the space above it should be filled with some dry, inert gas, such as nitrogen or carbon dioxide, having a pressure of one atmosphere at 300° , of four and one-half atmospheres at 450° , and of twenty atmospheres at 550° .

(6) Care must be exercised that a thermometer is not overheated. If a long portion of the stem is cold the stem correction may amount to 30° or 40° , and in this case the reading might be 500° when the temperature of the bulb was 540° and after a few minutes at that temperature the ice point might be 20° too low.

8. SUGGESTIONS.

The annealing oven, represented in Fig. 2, could be readily enlarged and adapted for use for annealing a large number of thermometers at a time. With the present small oven used on the ordinary incandescent lighting circuit the cost of annealing thermometers for six days and cooling for three days slowly should not exceed 50 cents each. This is a small expense since it represents the difference between a reliable instrument which will remain nearly constant and one which will show a change of several degrees after each heating. With a larger oven the cost for each instrument could be cut down at least half of the amount mentioned. Fig. 8 represents diagrammatically a regulating rheostat which could be operated automatically by means of a clock, or by hand, for gradually cooling off the oven. It is intended that the resistance of the coil between each pair of sections shall be so adjusted, by observing the resistance necessary to maintain, say, three given temperatures, that when the contact arm is set on any given section the temperature of the oven will remain at the point indicated by the number on that section. The slow cooling is not of great importance for thermometers which are not intended to be read closer than one degree or so, but a regulating rheostat would still be useful in set-

ting the temperature to the point desired. A similar oven could be designed for heating with gas and the cost might be slightly less than for an electrically heated oven, but the ease and certainty with which electric heating may be regulated and its constancy make it far more satisfactory.

Since one of the great advantages of mercurial thermometers is the direct reading scale, it is important that the corrections should be made as small as practicable. For this purpose the variations of the temperature scale defined by the different kinds of glass from

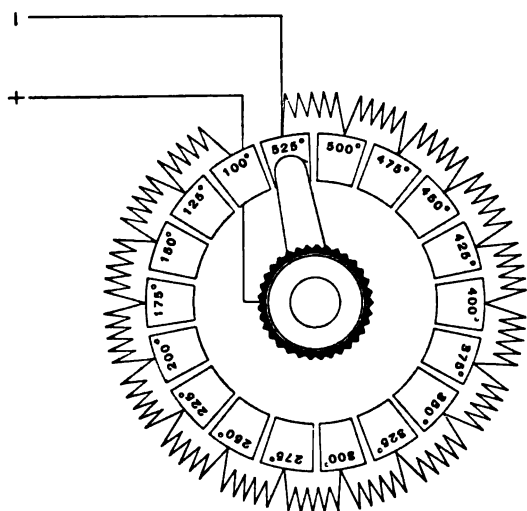


Fig. 8.—Regulating Rheostat.

the standard gas scale can advantageously be allowed for by the maker. These differences are given for two kinds of glass, Jena 59^{III} and Jena 16^{III}, in Tables I and II of this paper. One method of making a thermometer scale which shall read very nearly gas scale temperatures at all points is as follows: Divide the stem throughout its entire length by means of an auxiliary scale into divisions corresponding to one one-hundredth of the volume of the fundamental interval. Then if the thermometer be of Jena 59^{III} glass, for instance, the point 527.8 on the auxiliary scale is to be marked 500° on the thermometer; the point 469.1 is to be marked 450°, and so on, according to the table for this kind of glass. If each of these intervals be divided into 50 equal parts, the greatest error on the scale due to this method of adjusting will be only 0°.5. If the points were taken every 25°, the error would be much smaller.

If it is desired that thermometers should read true temperatures when immersed to a given point, with the emergent stem at a tem-

the standard gas scale can advantageously be allowed for by the maker. These differences are given for two kinds of glass, Jena 59^{III} and Jena 16^{III}, in Tables I and II of this paper. One method of making a thermometer scale which shall read very nearly gas scale temperatures at all points is as follows: Divide the stem throughout its entire length by means of an auxiliary scale into divisions corresponding

perature different from that of the bulb, the amount of this stem correction may be computed approximately by means of the following formula:

$$\text{Stem correction} = 0.00016 \times n \times (T - t) \text{ C}^\circ.$$

$$\text{Stem correction} = 0.000088 \times n \times (T - t) \text{ F}^\circ.$$

Where n = number of degrees emergent from the bath;

T = temperature of the bath;

t = mean temperature of the emergent stem.

The mean temperature, t , of the emergent column may be approximately measured by means of a small auxiliary thermometer suspended near it, or for lower temperatures by surrounding the stem with a small water jacket, and taking the temperature of the water with the auxiliary thermometer, or, more accurately, in the way suggested by Guillaume, who exposes a capillary mercury thread beside the stem and thus measures its mean temperature by the expansion of the mercury thread. This is even more conveniently carried out by the Faden thermometer of Mahlke,¹⁴ in which the expansion of the mercury in the long capillary (bulb) is measured by means of a still finer capillary stem. The corrections thus obtained can be combined with Tables I or II and a thermometer made to read nearly true gas scale temperatures under a given condition of immersion. For example, if the stem of a Jena 59^{III} glass thermometer be immersed to the 100° mark, at 400° the correction due to the emergent stem (assuming 50° for its mean temperature) will be $300 \times 350 \times 0.00016$ or 16°8. At 400°, according to Table I, this thermometer if divided into degrees of equal volume would read 412°3, but owing to the cooler stem it will read lower than this by 16.8 gas-scale degrees or 18°6 on the glass scale.¹⁵ The reading at 400° would therefore be 393.7 and this point is to be marked 400°. Other points can be determined in the same way.

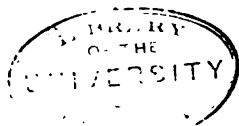
¹⁴ Mahlke, *Zs. für Instrumentenkunde*, 18, p. 58; 1893.

¹⁵ The stem correction computed in this way is expressed in gas-scale degrees and should strictly be reduced to degrees on the glass scale before using it, as above. In the present example the interval 350° to 450° on the gas scale comprises 469.1–358.1 or 111 glass-scale degrees, hence one gas-scale degree equals 1.11 glass-scale degrees and 16°8 on the gas scale equals 18°6 on the glass scale. The errors of an assumed stem correction are so large, however, that reducing to glass-scale degrees might often be omitted.

Thermometers after repeated heating and cooling even at moderate temperatures as in ordinary use show a decided depression of the ice point not corresponding to any of the effects discussed above. No reliable data have yet been obtained on this point, but it is hoped that an investigation may be made and an explanation found for this apparently anomolous change in glass.

In conclusion the author wishes to acknowledge his obligation to Dr. C. W. Waidner, who proposed the present investigation, and to Dr. G. K. Burgess, both of whom have made many valuable suggestions, and to Mr. J. R. Cain, of the chemical division, for a careful analysis of glass from one of the experimental thermometers.

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A NEW POTENTIOMETER FOR THE MEASUREMENT OF ELECTROMOTIVE FORCE AND CURRENT.

By H. B. Brooks.

1. INTRODUCTION.

Instruments for the measurement of electromotive force and current may be divided into two main classes. In those of the first class, which may be called deflection instruments, a movement of one element of the instrument is produced against an opposing force, an indication of the quantity to be measured being given by the extent of the motion, as shown by an index moving over a graduated scale. To this class belong many types of ammeters and voltmeters, such as the moving coil, the magnetic vane, electro-dynamometer, hot wire, and electrostatic instruments. Those of the second class may be called balance instruments. In these the electric quantity to be measured exerts a force which is opposed by another force, an adjustment of the instrument being made until no motion results, when the quantity to be measured is determined by the extent of the necessary adjustment. To this class belong Wheatstone bridges, potentiometers, and Kelvin balances.

Each of these two classes of measuring instruments has its advantages and disadvantages. The deflection instrument can not be read, as a rule, with an accuracy greater than one part in a thousand; while on account of zero errors, inaccuracy of the scale, heating due to the current, changes of external temperature, stray magnetic fields, and other disturbing influences, the accuracy really obtained in the result is much lower than this. It is possible, of course, to determine and allow for some of these errors, but others, such as those due to stray field, are not easily allowed for. The balance instrument can usually be read with greater accuracy, and in the

case of the potentiometer the accuracy of reading may be carried to almost any desired extent by using a sufficiently sensitive galvanometer.

Instruments of the deflection type are, as a rule, more liable to change their constants with time than those of the balance type. This is especially true of the moving coil instrument, where magnet and spring are subject to changes which affect the readings. Besides a gradual change in the strength of the magnet as time goes on, and sudden changes due to blows, a temporary change in the strength of the magnetic field in which the coil moves may be produced by heavy currents in conductors near which the instrument may be used or by the stray field from other instruments. Balance instruments, on the other hand, usually retain their calibration very well, and are not affected, either temporarily or permanently, by disturbing influences which seriously affect the other class.

Deflection instruments are especially useful in commercial work where great accuracy is not required. They have the advantages of requiring no special setting up, of being quickly read, and of always indicating, approximately at least, the value of the quantity to be measured. Balance instruments are particularly useful in laboratory work where accuracy is essential, and where steady electrical conditions can be maintained. Their use requires much time, as every change in the quantity under measurement requires an adjustment to be made for the condition of no deflection. In the determination of unknown masses, first by a spring balance and second by the equal arm balance, using standard masses, we have conditions similar to those attending the use of the two classes of electrical measuring instruments.

2. THE POTENTIOMETER METHOD.

For precision measurements of direct current and electromotive force, in terms of standards of electromotive force and of resistance, the potentiometer method excels in accuracy, flexibility, and range of application, and it is to-day the standard method for such measurements. The potentiometer is unaffected by external disturbances, if we except the effect of leakage currents, which should be guarded against. Its accuracy is dependent upon the accuracy of adjustment of resistance coils, and its permanency of calibration depends upon the

constancy of the relative values of these coils, and the constancy of the standard cell used. Resistances may be adjusted and compared with an accuracy much greater than that obtained in other electrical measurements, and the errors in the coils may be determined and allowed for at any time in accurate work. Standard cells, while not in a perfectly satisfactory condition at present, are readily compared with each other with great accuracy, and when properly made are very reliable.

On the other hand, the disadvantages of the potentiometer method are the time required to make the measurements, the bulk and cost of the apparatus, and the fact that the current or voltage must be kept steady while being measured, it being practically impossible to use a potentiometer on a circuit where the current or voltage is subject to fluctuations of any magnitude. Attempts have been made from time to time to introduce potentiometers for use as standard instruments in central station and similar work. An interesting example is the Howell voltmeter, used in early Edison stations. This was a portable potentiometer in compact form, comprising galvanometer, standard cell and resistance coils, one of these coils having a sliding contact, the position of which, when the galvanometer showed no deflection, was an indication of the pressure at the instrument terminals. One form of this instrument was small enough to be carried in a coat pocket. Special Daniell cells were used at first, and later a form of Clark cell, a thermometer being necessary with the latter.

Attempts have been made from time to time to introduce in practice more elaborate forms of portable potentiometer as standard instruments for use in checking deflection instruments; but these attempts have not met with entire success. The need unquestionably exists for a type of instrument for current and voltage measurements which will have properties intermediate between those of the ordinary deflection instrument and the elaborate potentiometer. For example, in reading ordinary portable ammeters and voltmeters the limit of accuracy of reading, for, say, two-thirds of full scale deflection, is about one part in a thousand. For checking these a standard instrument which can be easily read to this degree of accuracy and estimated five times further is amply sensitive. To meet this need there has been offered a larger form of deflec-

tion instrument for current and voltage measurements, to which the name "laboratory standard" has been applied. While the larger dimensions allow the heat generated to be thrown off with a smaller rise of temperature, and the longer pointer allows a somewhat more accurate reading of the deflection to be made, these instruments are not entirely satisfactory for the purpose for which they were produced, and for careful work need to be checked by a potentiometer at rather frequent intervals, under the conditions of use. These instruments are affected by stray field and other disturbing influences in the same way as the smaller instruments, though not to as great an extent.

The standard instrument should however be practically free from the sources of error which affect the deflection instrument, and should be so constructed that its constants will not be liable to alter with time. It should be possible to use it even on circuits that are not steady, and it should require as little manipulation as possible. It should be compact, simple, and not too expensive, and it should be made so as to be readily checked. In other words, it should combine as much as possible of the convenience of the deflection instrument and the accuracy and constancy of the potentiometer. Such an instrument would be useful in the central station, for the checking of portable and of switchboard instruments, some forms of which latter should be tested in position under the conditions of use. It would be of service in the instrument factory, where a large number of instruments must be tested, and in the laboratory, for measurements in which a greater degree of accuracy is required than can be had using deflection instruments. It would be of service in photometry, where a degree of accuracy in voltage measurements is needed which is obtained with deflection instruments only by great care in reading and by constantly falling back upon the potentiometer as a check on the relatively large changes, due to heating and other causes, which occur in the best deflection instruments. Other uses might be mentioned, but it is evident from the above that a field exists for an intermediate type of instrument.

3 THE DEFLECTION POTENTIOMETER.

With the above requirements in mind, it occurred to me that a satisfactory instrument for this work might be realized by using the potentiometer method to measure the bulk of the electrical

quantity, in conjunction with a suitable deflection instrument to measure the small remainder, which may be of the order of 1 per cent of the total.¹ For example, if an electromotive force is between 110 and 111 volts, the potentiometer method may be used to measure the 110, the decimal part being read by deflection. The potentiometer is essentially a null instrument, and the function of the galvanometer ordinarily is to show, by the absence of a deflection, when the unknown electromotive force is compensated by the known. In some cases, however, as in measuring low electromotive forces, we desire to read one more figure in addition to that given by the lowest dial; and in such a case, with some forms of potentiometer, we may interpolate between adjacent settings on this dial, just as we interpolate in Wheatstone bridge and other resistance work. In doing this we assume that within this narrow range the deflection is proportional simply to the unbalanced portion of the electromotive force. In the present case the problem is to determine the conditions to be met, first, in order that the deflection corresponding to a given unbalanced electromotive force may always be the same, regardless of the setting of the potentiometer dials; and, secondly, in order that the deflection should be strictly proportional to the unbalanced electromotive force for a relatively large unbalancing, as it is ordinarily assumed to be for a small unbalancing.

To reduce the necessary manipulation to a minimum, there should be but one dial to be set, one step of which covers from 1 to perhaps 5 per cent of the total range. In the latter case, to secure an accuracy of one-twentieth of 1 per cent, the deflection instrument must measure the unbalanced 5 per cent with an accuracy of 1 per cent. If the steps on the main dial are not so large, the accuracy required in the galvanometer is not so great.

¹ The idea of balancing the larger portion of an unknown emf. by the potentiometer method, the remainder being indicated by the deflection of a galvanometer, was suggested by Stansfield and used by him in the construction of a recording pyrometer. (*Phil. Mag.* 5, 46, p. 59, 1898.) In his apparatus a Clark cell of large size was used to furnish the working current through the potentiometer wire, and it was necessary to have a second potentiometer to determine the value of this current. More recently the same idea has been used for pyrometric work by Hoffmann and Rothe. (*Zs. für Instrumentenkunde*, Sept., 1905.) In their apparatus also the indication of the galvanometer does not give at once the value of the unknown emf., auxiliary observations and calculations being required.

4. THEORY OF THE DEFLECTION POTENTIOMETER.

Assuming that the instrument is to be used for measuring electromotive forces higher than those of a single cell, a "volt-box" will be required, as shown in Fig. 1, in which E is the unknown emf. to be measured, e_1 that of the auxiliary storage cell, R the total resistance of the volt-box, the fall of potential around a fraction of this, R/p , being opposed to the fall of potential around a portion, r_1 , of the potentiometer wire AB . It was seen that to obtain the condition of constant sensibility a rheostat would be required in the galvanometer circuit, this rheostat being controlled by the motion of the main dial, represented by the slider S . For simplicity the standard cell and connections for using it to check the working current in AB are not shown, as they do not affect the problem. Assuming that the condition of balance does not exist, and denoting the currents and resistances in the various branches by $i_1, i_2, \dots, r_1, r_2, \dots$ and applying Kirchhoff's laws, we have the following equations:

$$i_1 - i_2 + i_g = 0 \quad (1)$$

$$i_2 - i_3 - i_g = 0 \quad (2)$$

$$i_1 r_1 + i_2 (r_2 + r_3) = e_1 \quad (3)$$

$$i_1 r_1 - i_2 (r_2 + r_3) - i_g \frac{R}{p} = 0 \quad (4)$$

$$i_2 R \frac{p-1}{p} + i_g \frac{R}{p} = E. \quad (5)$$

Solving these equations, we get

$$i_g = \frac{e_1 \left(\frac{r_1}{r_1 + r_2 + r_3} \right) - \frac{E}{p}}{r_1 + r_2 + \frac{r_1 (r_2 + r_3)}{r_1 + (r_2 + r_3)} + R \frac{p-1}{p^2}} \quad (6)$$

The first term in the numerator of this expression is the fall of potential in the portion r_1 of the potentiometer wire when the galvanometer circuit is open; it is therefore numerically equal to the setting of the potentiometer. The second term in the numerator is the fall of potential which would exist around the portion R/p of

the volt-box if the galvanometer circuit were open. The denominator is the total resistance in the galvanometer circuit, the third term being the resultant resistance of the portion r_1 of the potentiometer wire shunted by the remainder of the battery circuit, and the fourth term is the resistance of the portion R/p in parallel with the remainder of the volt-box, $R \frac{p-1}{p}$. Equation (6) therefore shows that the current through the galvanometer is equal to the unbalanced portion of the electromotive force divided by the total resistance of the galvanometer circuit; or

$$i_g = \frac{\Delta e}{\Sigma r} \quad (7)$$

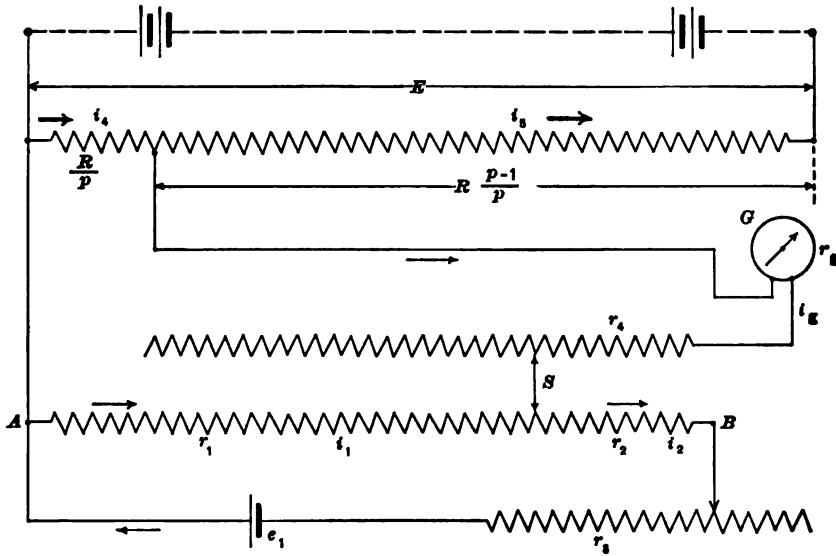


Fig. 1.—First Plan of Circuits.

Referring to equation (6), we may denote the first term in the numerator, which may be called the setting, by s . Since the volt-box has a multiplying power of p , the equation may be written in the form

$$i_g = \frac{ps - E}{p \Sigma r} \quad (8)$$

which shows that if Σr can be kept constant the galvanometer current will be directly proportional to $(ps - E)$, the difference between

the emf. corresponding to the setting and the emf. to be measured. If Σr can be kept constant for all settings, it is only necessary to calibrate the scale of the galvanometer properly to make it read directly the unbalanced part of the emf. under measurement. Referring to Fig. 1 and equation (6), it will be seen that the resistance r_4 must have such values for different positions of the slider S that the sum of r_4 and $\frac{r_1(r_2+r_3)}{r_1+(r_2+r_3)}$ will be a constant. The latter has a maximum value at some point between A and B , and minimum values at A and B ; so that r_4 must vary accordingly.

A difficulty in the way of using this particular arrangement of circuits lies in the fact that the value of e_1 depends on the condition of the storage cell, and as e_1 varies r_3 must be varied to keep the proper current through AB . This double variation of the setting and of r_3 is difficult to compensate for accurately, since for settings near A changes of r_3 make very little change in the resultant resistance, while for settings near B the changes of r_3 enter almost undiminished into the resistance of the galvanometer circuit. We may use a number of cells in place of one, giving a larger current through AB , and thus limiting the part of AB used to a relatively small portion near A . This would give a sufficiently accurate compensation for most purposes; but another arrangement of circuits may be used which does not have the objection of requiring a number of cells, one being sufficient, while the compensation for changes in the setting and in the battery rheostat may be made as perfect as desired. The plan of circuits is shown in Fig. 2.

In this arrangement the setting is made on the volt-box instead of on the potentiometer wire. The rheostat r_3 is set so as to give, by reference to a standard cell (not shown), a certain standard current through AB , and therefore a constant fall of potential around it. This fall of potential is balanced as nearly as possible by setting the slider S . The variable resistance r_4 is arranged so that r_4 plus the resultant resistance of the potentiometer circuit is a constant; r_4 thus takes care of changes in resistance required by variations in the emf. of the auxiliary cell, but is independent of the setting. The fraction $\frac{I}{P}$ being in this case a variable, the resistance in the galvanometer circuit must vary with it, as may be seen from eq. (8).

Since in this second arrangement the emf. term $e_1 \frac{r_1}{r_1 + r_s}$ is a constant, we may denote it by e . We have then

$$i_g = \frac{1}{p} \cdot \frac{pe - E}{\Sigma r} \quad (9)$$

For a difference of one volt between the setting pe and the emf. E under measurement the galvanometer is to give a deflection of m

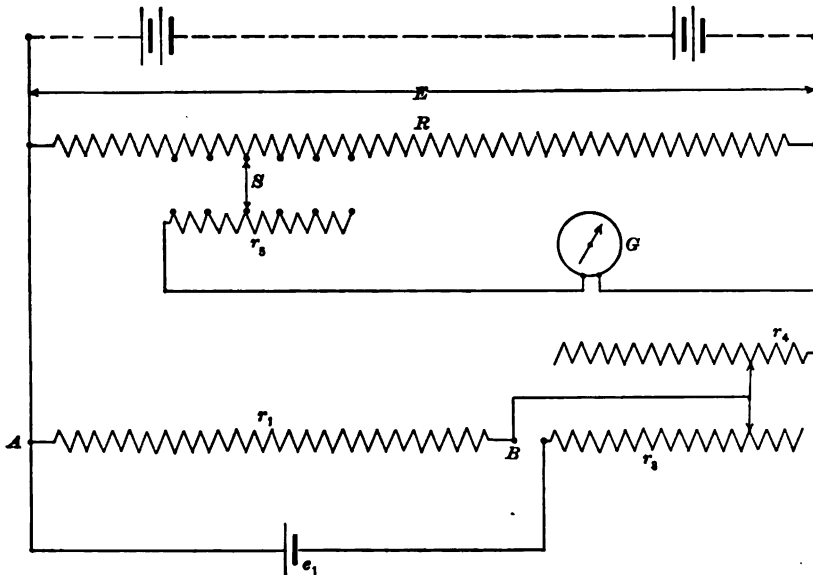


Fig. 2.—Second Plan of Circuits.

scale divisions. If I denote the current required to give a deflection of one scale division, the current mI must always flow when $pe - E = I$. Substituting these values, we have

$$mI = \frac{1}{p} \cdot \frac{I}{\Sigma r} \quad (10)$$

$$\Sigma r = \frac{I}{pmI} \quad (11)$$

Eq. (10) shows that the compensating resistance r_s in the galvanometer circuit, which is controlled by the movement of the slider S , must vary so as to keep the product $p \Sigma r$ a constant. Eq. (11) holds

for the first arrangement also, shown in Fig. 1; in this case, ρ being a constant, Σr must be constant and equal to $\frac{1}{\rho m I}$.

The galvanometer to be used as a part of this instrument should be of the needle type, preferably one that is pivoted and does not require an arrestment. It should have a good zero, a short period, and a calibrated scale. For laboratory use a reflecting galvanometer of short period may be used, but in many applications, as in checking switchboard instruments in position, the use of such a galvanometer would be a serious drawback, and since the high sensibility of the reflecting galvanometer is not required for the middle class of measurements for which this type of potentiometer is designed, it would seem better to use, in most cases, the more compact and easily manipulated needle instrument, which makes the apparatus portable and always ready for use. The galvanometer should have, for use in voltage measurements, a sensibility of a millimeter deflection for from two to four microamperes, unless a very high resistance is required in the volt-box circuit. The calibration of the scale should be carefully tested, as there are portable galvanometers which have fair working qualities and high average sensibility, but which have a sensibility at some parts of the scale nearly double that at others. The satisfactory working of the apparatus will depend considerably upon the working qualities of the galvanometer, and hence care should be used in selecting this instrument. A point to be considered is the value of the total resistance at which the galvanometer is aperiodic. For good results the resistance in the galvanometer circuit should be equal to or slightly greater than this resistance. In the first arrangement of circuits (Fig. 1) in which the total resistance of the galvanometer circuit is a constant, the damping may be external to any desired extent; but for the second case, with a large variation in the resistance of the galvanometer circuit, this resistance should be relatively high and the damping largely internal.

5. CONSTRUCTION OF THE INSTRUMENT.

A deflection potentiometer has recently been constructed in the instrument shop of this Bureau from my designs in accordance with the foregoing principles, and has been put into service for voltage measurements in the photometric work of the Bureau.

In this work a standard potentiometer has been used for the most accurate measurements, and it will continue to be used. The less accurate measurements had previously been made with regular portable instruments, but it was found impossible to secure the desired accuracy in this way. Portable instruments must be frequently calibrated, and must be connected to the circuit at the proper voltage for some time before use, in order to warm up and reach a steady condition. The estimation of tenths of a division requires close inspection and a relatively large amount of time, while the personal equation enters into the result. The measurement of voltage as previously carried out set a limit of accuracy to the photometric measurements somewhat lower than was desired. A potentiometer of standard form would give even greater accuracy than is required, but its use would consume too much time. The deflection potentiometer gives ample accuracy, the estimation of tenths of a division being unnecessary, and in using it for a lot of lamps of approximately equal voltage accurate readings can be taken more rapidly than with a voltmeter. If every lamp has a different voltage, the necessary adjustment of the dial would reduce the speed, but even in this case the consideration of accuracy in the work of the Bureau is given greater weight than the speed of working.

The range of this instrument is from 95 to 125 volts. Fig. 3 gives a diagrammatic plan of the circuits in the instrument. The main dial has 15 steps, each of 2 volts, the galvanometer normally covering a range of 2 volts, although it may be used over a range of 3 volts; that is, 1.5 volts on each side of the setting. To avoid errors in the photometric work the resistance R was made high, being 50,000 ohms. The resistance in the battery circuit is 2,000 ohms total when the emf. of the storage cell is 2 volts. On account of these high resistances the galvanometer sensibility required is greater than would ordinarily be necessary. The galvanometer was supplied by the Weston Electrical Instrument Company and is their regular model A, of higher sensibility than usual, giving 1 mm deflection for 1.6 microamperes. The internal resistance is 270 ohms and the period on open circuit 1.3 seconds. When in use, with its circuit closed through the potentiometer, the motion of the needle is very nearly aperiodic and readings may be quickly made.

The battery current is regulated by a rheostat of 10 steps in con-

nection with a slide rheostat for fine adjustment. The emf. of the storage cell may have any value between 1.9 and 2.1 volts. These rheostats, both dial and slide, are double, so that as resistance is added in the battery circuit it is cut out of the galvanometer circuit. A four-point switch is used to connect the galvanometer to the emf. circuit, to the standard cell through a high resistance, or to the standard cell directly. The remaining point opens the galvanometer circuit. A contact key is also provided for momentarily closing the galvanometer circuit before the balance is close enough to

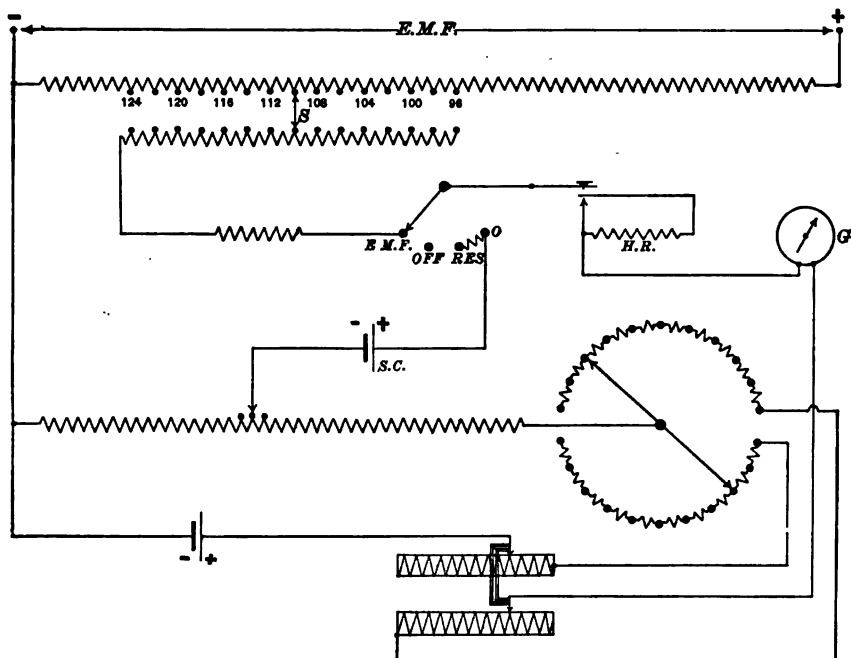


Fig. 3.—Connections of Potentiometer. (Diagrammatic.)

read the deflection. This key has an intermediate high resistance contact, as indicated in Fig. 3, and may be locked down.

A view of this deflection potentiometer is given in Fig. 4, from which it is seen that the galvanometer is built into the apparatus, so that it is self-contained, with the exception of the standard cell and storage cell. If the instrument had been intended for portable use, as in checking switchboard instruments in position, the storage cell could have been replaced by two dry cells, which with the

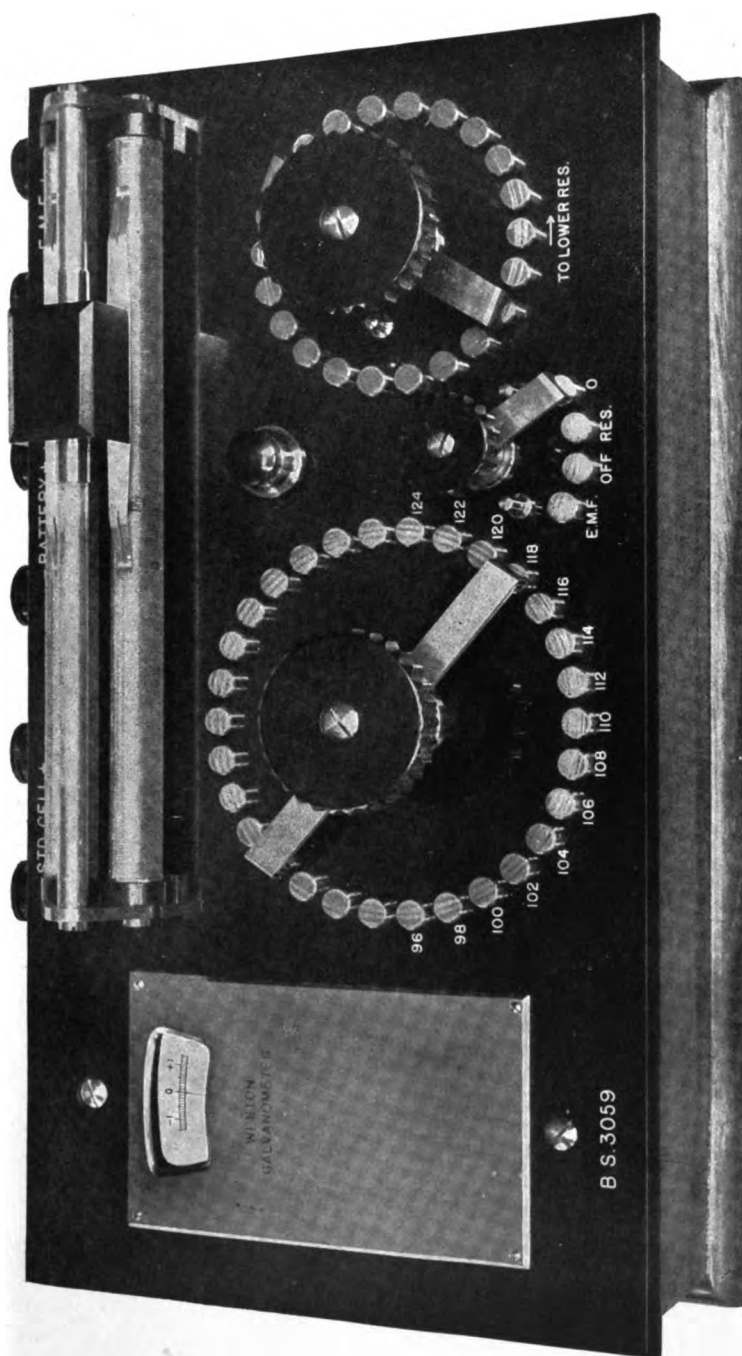


Fig. 4.—Deflection Potentiometer (Half Size).

standard cell could have been inclosed in the case, making the entire apparatus self-contained. Special precaution was taken to guard against leakage from the volt-box coils to the low-voltage circuits and the galvanometer, the guard-ring principle being used. The dimensions of the instrument are as follows: Length, 33 cm; breadth, 19 cm; height, 13.6 cm. The weight without cells is 4.75 kilograms.

A test showed that over the whole range the greatest error was 0.02 volt, the error when no deflection occurs, as at 96, 98, etc., not exceeding 0.01 volt at any point.

The design of a larger instrument of this type is under consideration. This will be used for checking voltmeters, and the dial will consequently have a greater number of points than in the instrument just described. By means of a switch the standard cell may be balanced around different portions of the potentiometer wire, giving working currents through the wire in proportion to the numbers 1, 2, 4, or whatever values may be desired. By making the volt-box so that the highest voltage to be measured will not produce undesirable heating, the same instrument may be used for various ranges, as 150, 300, and 600 volts. The details of this second instrument are not sufficiently developed at this time to give a description of it.

6. APPLICATION TO CURRENT MEASUREMENT.

It is interesting to note that while the plan of circuits shown in Fig. 1 is not convenient for use in measuring voltages higher than those of a single cell, it is the most suitable one to use in a deflection potentiometer for measuring currents. Here the fall of potential at the terminals of a current shunt is to be measured, and this fall of potential being small, it is desirable to use all of it. Further, if we were to use the plan shown in Fig. 2 for this work it would be necessary to construct a special shunt for each range, with a large number of potential points accurately located. With the plan shown in Fig. 1 any standard current shunt may be used, and it need not be an integral part of the potentiometer proper, which thus constitutes a precision millivoltmeter.

The circuits for an instrument of this kind are shown in Fig. 5. A current shunt W replaces the portion $R \rho$ of the volt-box in Fig. 1,

and only a limited portion of the potentiometer wire AB is used, corresponding to the drop in the shunt at full load, say 150 millivolts. Since only a small portion of the wire AB is used, the compensating resistance r_1 may correct for the small changes in the resultant resistance as the battery rheostat r_2 is altered, for the position of S corresponding to full load through the shunt. The compensation for smaller loads will not be theoretically exact, but the error may be made negligibly small. In an instrument for this purpose it is an advantage to use a smaller resistance in AB , in order to reduce the necessary sensibility of the galvanometer and to keep within limits the source of error just referred to. By proper design a high

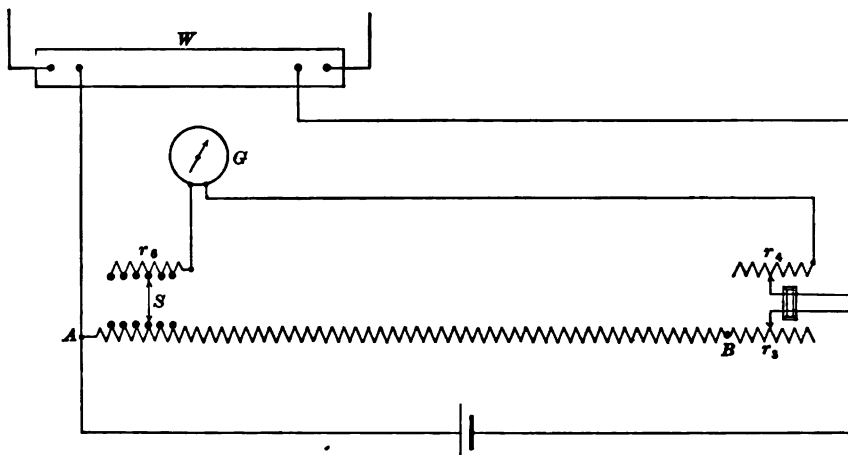


Fig. 5 —Plan of Circuits for Current Measurement.

grade of portable galvanometer like the one before mentioned, if suitably wound, will be sensitive enough, and will be of advantage when the instrument is to be used, say, for checking ammeters in position on switchboards and for use in laboratories when it is desired to have a portable precision ammeter which is independent of external disturbing forces. In heavy current tests it is usually difficult to maintain the current steady, and for such work a deflection potentiometer would be very useful, as the value of the current may be read at any moment on the standard instrument and the one under test.

SPECTRUM LINES AS LIGHT SOURCES IN POLARISCOPIC MEASUREMENTS.

By Frederick Bates.

Recent progress in polariscopic work necessitates careful consideration of the character and behavior of the light sources used. Unless the source can be accurately defined and its constancy depended upon, it is obviously impossible to compare the polariscopic measurements of different observers.

In the half-shade polariscope the circular field is divided into two equal parts by a black line. When a setting of the instrument is

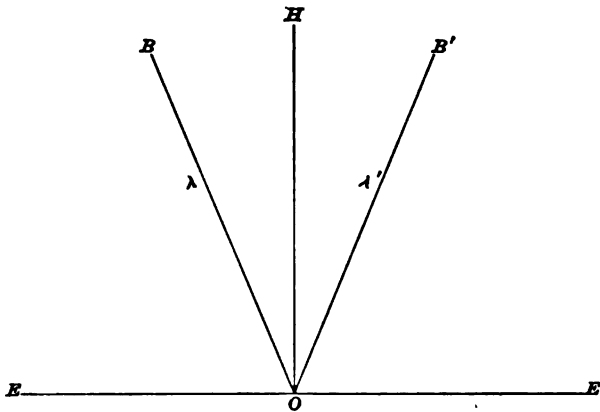


Fig. 1.

made the analyzer is rotated into a position which makes the illumination of the two halves equal. This condition is represented in Fig. 1.

The normal HO to the polarization plane (EE') of the analyzer bisects the angle BOB' where BO and B'O are the traces of the polarization planes of the large and small nicols of the polarizing

system. If the source be monochromatic and a rotation takes place anywhere between the polarizing system and the analyzer, HO again bisects BOB' when the condition of equal illumination is established. If the source be not monochromatic, only the polarization planes of the waves with the same wave length coincide after the rotation occurs.

Let a source, as shown in Fig. 2, consist of two different wave lengths, one of which is λ_1 and λ'_1 in the left and right halves of

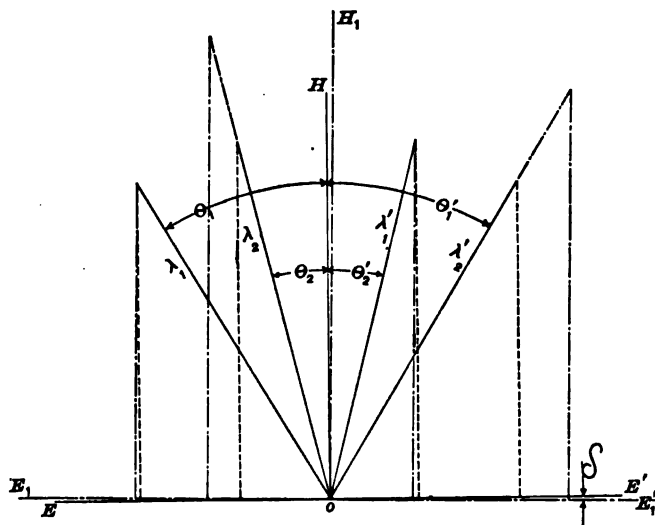


Fig. 2.

the field respectively; and the other similarly λ_2 and λ'_2 . Let the intensities of λ_1 and λ'_1 be A_1 , and of λ_2 and λ'_2 be A_2 . When EE occupies the position for equal illumination we have

$$A_1 \sin^2 \theta_1 + A_2 \sin^2 \theta_2 = A_1 \sin^2 \theta'_1 + A_2 \sin^2 \theta'_2 \quad (1)$$

in which the right and left hand members are the intensities of the corresponding halves of the field as seen through the analyzer.

$$\begin{aligned} \text{Let} \quad \theta_1 &= \phi_1 \pm \delta & \theta_2 &= \phi_2 \pm \delta \\ \theta'_1 &= \phi_1 \mp \delta & \theta'_2 &= \phi_2 \mp \delta \end{aligned}$$

Equation (1) becomes

$$A_1 \sin^2(\phi_1 \pm \delta) + A_2 \sin^2(\phi_2 \pm \delta) = A_1 \sin^2(\phi_1 \mp \delta) + A_2 \sin^2(\phi_2 \mp \delta) \quad (2)$$

Solving (2) for δ by substituting

$$\frac{A_2}{A_1} = K \text{ and } \sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$$

we obtain

$$\tan 2\delta = \pm \frac{K-1}{K+1} \tan (\phi_1 - \phi_2) \quad (3)$$

or since all the angles are small

$$\delta = \pm \frac{1}{2} \frac{K-1}{K+1} (\phi_1 - \phi_2) \quad (4)$$

When $K=1$ we have

$$\delta=0; \theta_1=\theta'_1; \theta_2=\theta'_2$$

Equation (2) then expresses the condition for equal illumination when measuring a rotation for a wavelength which is the so-called optical center of gravity of the two wavelengths when they have equal amplitudes.

Only in rare instances is K unity. Thus δ is a rotation which is the difference between the rotation when the intensities corresponding to the two wavelengths are equal, and the rotation when they are unequal. In general, when the shorter of the two wavelengths has the greater intensity the sign of δ is positive, and it is in the same direction as the rotation of the substance. By means of (4) we easily calculate the change in the rotation for any variation in the intensity of either wave. With a given line structure and K constant, δ is a function of $(\phi_1 - \phi_2)$ only, which in turn is a function of the rotation dispersion curve. With certain substances showing anomalous rotation dispersion the sign of $(\phi_1 - \phi_2)$ might be reversed. Hence the assumption which has long been accepted as a fact¹ that the optical center of gravity is independent of the rotation dispersion is incorrect.

The angular rotation δ resulting from increasing the intensity of one of the waves is shown graphically in Fig. 2.

¹See Landolt, *Das Optische Drehungsvermögen*, 2 Auflage, p. 360.

In the case of the two spectral lines of incandescent sodium vapor used as a source, for a thickness of quartz giving a rotation of about 253°

$$\begin{aligned}\lambda_1 &= 589.62 & A_1 &= 1 \\ \lambda_2 &= 589.02 & A_2 &= 2 \\ (\phi_1 - \phi_2) &= 0.466^\circ\end{aligned}\tag{5}$$

Substituting in (4) $K = 1.6.$

$$\delta = \frac{1.6 - 1}{2(1.6 + 1)} 0.466 = 0.054^\circ.$$

The total rotation measured is therefore 0.054° greater than that which would have been obtained with $\lambda = 589.32 \mu\mu$ (mean of $\lambda_1 + \lambda_2$), which gives $589.25 \mu\mu$ as the optical center of gravity of these lines for all substances having approximately the same rotation dispersion as quartz.

Owing to the fact that they are easily obtained quite intense and free from the presence of other lines, the two spectral sodium lines $\lambda = 589.62$ and 589.02 have been very extensively used as a monochromatic source. All of the optical constants used in polariscopic work have been determined with them. After an extensive investigation the writer has found no sodium source that is an improvement on sticks of fused Na_2CO_3 fed into an oxyhydrogen flame. However, the lack of intensity and unstable line structure under certain conditions renders even this source far from satisfactory, owing to the great precision required in polariscopic measurements. In utilizing this source the rod of fused Na_2CO_3 is placed in the flame at one of the positions shown in Fig. 3.

Through the kindness of Dr. P. G. Nutting it was possible to examine the line structure of the sodium lines as well as that of the line $\lambda = 546.1 \mu\mu$ of mercury vapor with an echelon spectrocope. Table I refers to Fig. 3.

The lines obtained by this method are extremely sharp and differ from the arc spectra by the absence of the characteristic haziness of the edges. If displacements as large as $0.5 \mu\mu$ had occurred, they would have been readily detected. The broadening was at all times symmetrical. In this respect the observations agree with the more recent work of other observers, but are contradictory to the results obtained by Ebert² which have been accepted in polariscopic work.

²Ebert: Wied. Annalen, 84, 39 (1888).

By careful manipulation of the flame in position 3 reversal of the lines took place. D_2 preceded D_1 in this respect.

It is thus evident that so far as their line structure is concerned

TABLE I.

Structure of Sodium Lines at Different Intensities.

	Relative Intensity	Position 1, Width $\mu\mu$	Position 2, Width $\mu\mu$	Position 3, Width $\mu\mu$		
				Na_2CO_3 Melting Slowly	Na_2CO_3 Melting Very Rapidly	
D_1	1	0.008	0.02	0.3 to 0.4	0.4	Reversed 0.6
D_2	1.6	0.008	0.02	0.3 to 0.4	0.6	Reversed 0.6

the sodium lines can be depended upon to give a sufficiently definite optical center of gravity up to the point of reversal in position 3. However, very noticeable variations in polariscopic measurements are likely to be observed with sodium sources at different intensities. These variations it is believed are not due to changes in the line structure of the source, but to the difficulty of excluding all extraneous light even with a very narrow slit and a dispersion sufficient to separate D_1 and D_2 . This extraneous light constitutes a different percentage of the total illumination whenever the intensity of the source varies. It may, therefore, appreciably affect the optical center of gravity when the field appears dim, and inversely when bright. Aside from the color and stability of the sodium lines up to reversal there is nothing in their favor as a polariscopic source. The flame requires the constant attention of an assistant, and even in reversal the intensity is not nearly sufficient to permit of the use of the greatest sensibility of a good polarizing system.

It is proposed to use the green line of incandescent mercury vapor $\lambda = 546.1 \mu\mu$ as the standard source for all accurate polariscopic work. Quartz mercury-vapor lamps as now made are reliable in

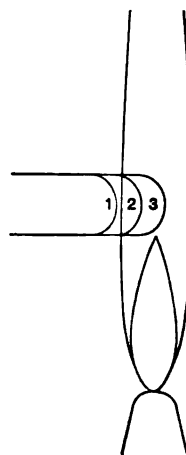


Fig. 3.

action. If sufficient care is exercised in preparing the mercury and exhausting the lamps, the five characteristic lines only will be obtained in the visible spectrum and with great intensity.

Different observers have found markedly different line structures for the line $\lambda = 546.1 \mu\mu$ depending upon the method of analysis employed. The map in Fig. 4 was obtained with the echelon with 1.8 amperes passing through the lamp. The fractional values given are the relative intensities as nearly as they could be estimated. It was found impossible to decide whether the satellite $-.024$ belongs to the positive or negative side of the primary. When the current was increased to more than 2.1 amperes the satellite $-.055$ increased in intensity until it about equaled the primary. The difference in wave lengths of the extreme satellites is less than $0.04 \mu\mu$. With a

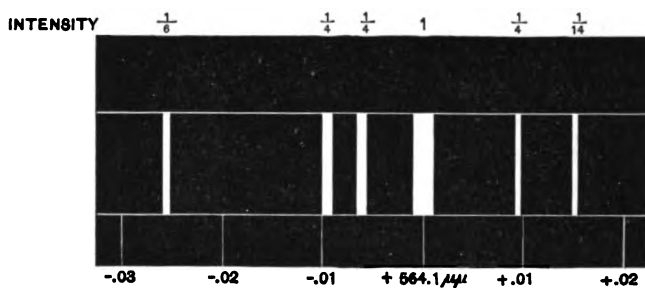


Fig. 4.

different source Fabry and Perot³ found $0.035 \mu\mu$ and Houstoun $0.0215 \mu\mu$. The distance between D_1 and D_2 is fifteen times $0.04 \mu\mu$. As far as it has been possible to determine, the line structure for the quartz lamp under widely varying conditions is such that for polariscopic purposes $\lambda = 546.1 \mu\mu$ is a monochromatic source of great intensity and perfect reliability. In measuring a rotation of two hundred and fifty degrees no differences could be detected due to changes in the emission, and probably none for much larger rotations. The quartz lamp requires little attention and can be operated indefinitely. Since only the lines $\lambda = 579.0, 576.9, 546.1, 435.8,$ and $404.7 \mu\mu$ are present the difference in wave length is such as to permit of perfect separation of the line $546.1 \mu\mu$ by even a relatively

³C. R., 1898, p. 409.

⁴Houstoun Phil. Mag. 7, p. 456 (1904).

small dispersion, and without bringing other lines in close proximity to the edges of the slit. Hence this source permits the elimination of practically all diffused light.

546.1 $\mu\mu$ by even a relatively small dispersion, and without bringing other lines in close proximity to the edges of the slit. Hence this source permits the elimination of practically all diffused light.

In order to make the advantages of the above system available, the temperature coefficient (a) of quartz for $\lambda = 546.1\mu\mu$ as well as the constant $\frac{\phi_{\lambda=546.1}}{\phi_{\lambda=589.2}}$ (where ϕ is a rotation) were determined. It was

thought advisable to measure a , although V. Lang,⁵ Sohncke,⁶ and Le Chatelier⁷ state that it has the same value for all wave lengths.

We have
$$\phi_t = \phi_0(1 + at)$$

where ϕ_t is the rotation at temperature t and ϕ_0 at zero. For any other temperature t we have

$$\phi_{t_1} = \phi_0(1 + at_1)$$

Hence

$$a = \frac{\phi_{t_1} - \phi_t}{\phi_{t_1}t_1 - \phi_t t}$$

The measurements were made with the most improved types of apparatus. $\lambda = 546.1\mu\mu$ was separated by means of two dense prisms. The left rotating quartz plate was placed in a water cell closed with glass end plates. The cell could be opened and the quartz removed without disturbing the mounting of these plates. This is of the greatest importance, as it insures constancy of the optical path at all times save that caused by the presence and absence of the quartz plate. Water at the desired temperature flowed through the glass cell during all measurements. Twenty consecutive settings were always made—five on the zero point, ten with the quartz inserted, and five on the zero. A summary of the data is given in Table II.

By combining the values of ϕ at the different temperatures from the data in Table II we obtain the following:

$$a = 0.000144 \pm 0.00000087$$

⁵ V. Lang: Pogg. Annalen 156,422 (1875).

⁶ Sohncke: Wied. Annalen 3,516.

⁷ Le Chatelier: C. R. 109-244 (1889).

Hence

$$\phi_t = \phi_0 (1 + 0.000144t)$$

between 4° C and 50° C.

In measuring $\frac{\phi_{\lambda=546.1}}{\phi_{\lambda=589.25}}$ a very large number of determinations were

TABLE II.

Rotation for $\lambda=546.1 \mu\mu$.

Observation No.	Temperature	Rotation ϕ
1	4°854 C	136°8968
2	6°230	136°9258
3	8°302	136°9714
4	16°049	137°0984
5	20°761	137°1958
6	29°575	137°3853
7	44°568	137°6832
8	49°799	137°7929

TABLE III.

Value of ϕ .

Observations used. Nos.	ϕ
1 and 5	0.0001377
2 " 6	0.0001439
3 " 7	0.0001436
4 " 8	0.0001505
1 " 7	0.0001447
1 " 8	0.0001481
2 " 7	0.0001444
2 " 8	0.0001430

made, practically all of which were concordant. However, in order to eliminate the personal equation and avoid as far as possible errors due to the character of the sodium source, the value of ϕ is computed from the measurements of five plates whose sodium value has recently been determined at the Physikalisch-Technische

Reichsanstalt. The mean of this value and that of the writer was taken as the rotation for $\lambda = 589.25$. The rotations were measured in part with a sensitive strip polarizing system. The greater number, however, were made with an exceptionally good Lippich system. The room temperature was held near to 20° C. The largest probable error given by the Physikalisch-Technische Reichsanstalt was $\pm .0006^\circ$; by the writer, $\pm .0004^\circ$. A set of readings for $\lambda = 546.1 \mu\mu$ on Plate No. 4, selected at random, is given in Table IV.

TABLE IV.

Readings on Quartz Plate No. 4.

Plate No. 4. Source $\lambda = 546.1 \mu\mu$.

Vernier A		Vernier B		T. Deg.
Plate in	Plate out	Plate in	Plate out	
208°888	176°364	28°893	356°366	20°90 C.
208°890	176°366	28°895	356°368	
208°889	176°358	28°893	356°359	
208°885	176°361	28°890	356°363	
208°890	176°364	28°895	356°366	
208°889	176°360	28°894	356°362	
208°885	176°358	28°890	356°361	
208°885	176°364	28°890	356°366	
208°885	176°367	28°890	356°370	
208°888	176°362	28°892	356°365	
				20°80

From the above

$$\phi_{\lambda=546.1} = 32^\circ.522 \text{ at } 20^\circ \text{ C.}$$

The data is summarized in Table V, wherein N. B. S. stands for National Bureau of Standards, and P. T. R. for Physikalisch-Technische Reichsanstalt.

We have therefore

$$\frac{\phi_{\lambda=546.1}}{\phi_{\lambda=589.25}} = 1.175166 \pm 0.000012 \quad (7)$$

$$\frac{\phi_{\lambda=589.25}}{\phi_{\lambda=546.1}} = 0.850944 \quad (8)$$

TABLE V.

Measurement of $\frac{\phi_{\lambda=546.1}}{\phi_{\lambda=589.25}}$

Plate No.	$\phi_{\lambda=589.25}; t=20^{\circ} \text{C}$			$\phi_{\lambda=546.1}; t=20^{\circ} \text{C}$	$\phi_{\lambda=546.1}$
	N. B. S.	P. T. R.	Mean	N. B. S.	$\phi_{\lambda=589.25}$
1	34.5978	34.596	34.5969	40.6581	1.17519
2	27.6547	27.644	27.6493	32.4912	1.17512
3	34.6292	34.626	34.6276	40.6930	1.17516
4	27.672	27.676	27.674	32.5220	1.17518
5	20.9169	20.916	20.9165	24.5806	1.17518

Thus any quartz rotation for the wave length $589.25 \mu\mu$ may be obtained by measuring the rotation for the wave length $546.1 \mu\mu$ and multiplying it by the constant 0.850944. By this method the errors due to the character of the sodium source of light are eliminated and the measurements of one observer may be readily compared with those of another.

POLARIMETRIC SENSIBILITY AND ACCURACY.

By P. G. Nutting.

This article is a theoretical investigation of polarimetric instruments and methods. Particular attention is given to the construction and use of the fundamental polarimetric equations, to the limits of accuracy with different forms of analyzer, and to the errors due to lack of homogeneity in the sources of light employed.

The fundamental equation of polarimetry is an expression for the light transmitted by a pair of nicols in terms of the angle between the nicols, the amount of light entering the first nicol and the rotation produced by a body inserted between the nicols. This equation is

$$I = I_0 \sin^2 (\theta - \rho) \quad \text{or} \quad I(\lambda) = E(\lambda) \sin^2 [\theta - \rho(\lambda)] \quad (1)$$

expressing as functions of the wave length (λ), quantities which depend upon the quality of the light used. The angle θ between analyzer and polarizer is measured from a crossed position of the nicols. I is the intensity of the light transmitted of the wave length, for which E is the intensity of the source used and ρ is the rotation under investigation.

The setting of the analyzing nicol (or nicols) is an absolutely independent variable, while I , I_0 , and ρ are complicated functions of wave length, or more properly, of wave period. Equation (1) holds for any particular wave period, and hence for all wave periods. The total light transmitted by a pair of nicols in any relative position is given by the integral of (1) with respect to wave period. The analyzing nicol is set at such an angle as to make the value of this integral a minimum in case the analyzer consists of a single

nicol. With a half shade analyzer the setting is such as to make the value of this integral equal to the value of a similar integral in which θ has been altered by a small constant angle.

The rotation function $\rho(\lambda)$ has been developed in the form

$$\rho = \frac{k_1}{\lambda^2 - \lambda_1^2} + \frac{k}{\lambda^2}$$

by Drude¹ and found to express the rotary dispersion of quartz over a wide range of wave lengths. The author² has developed this function in the form

$$\rho = \frac{k(\lambda^2 - \lambda_1^2)}{\lambda^2(\lambda^2 - \lambda_1^2 + a_1)},$$

which was found to represent the rotation of sugar and other solutions in the visible and ultraviolet regions with great accuracy. The function $I_0(\lambda)$ corresponds with the emission function $E(\lambda)$ for the source used. None of these functions have yet been constructed, but they appear to be modifications of the exponential function of the form

$$E = cf(\lambda)e^{\phi(\lambda)}$$

The general unlimited expression for the total amount of light transmitted by a pair of nicols is then the integral of (1) with respect to the wave length, or

$$T = \int I(\lambda) d\lambda = \int E(\lambda) \sin^2 [\theta - \rho(\lambda)] d\lambda. \quad (2)$$

Consider now the applications of equation (2) to the measurement of rotation with (1) a simple analyzing nicol, (2) a half shade analyzer.

1. ANALYZER CONSISTING OF SINGLE NICOL.

In this case the analyzing nicol is set at such an angle as to make the transmission T in equation (2) a minimum. This condition gives

$$\frac{dT}{d\theta} = \frac{d}{d\theta} \int E \sin^2 (\theta - \rho) d\lambda = 0;$$

¹ P. Drude: *Lehrbuch der Optik*, Leipzig, 1900, p. 381.

² P. G. Nutting: *Physical Review*, 18, p. 24, July, 1903.

hence

$$\begin{aligned} \int 2E \sin (\theta-\rho) \cos (\theta-\rho) d \lambda &= 0 \\ \int E \sin 2(\theta-\rho) d \lambda &= 0, \end{aligned} \quad (3)$$

from which

$$\tan 2 \theta = \frac{\int E \sin 2 \rho d \lambda}{\int E \cos 2 \rho d \lambda} \equiv \theta(E, \rho) \text{ say.}$$

This value of θ gives the setting of the analyzer for which the intensity of the transmitted light is a minimum for any sort of heterogenous source whose emission function is $E(\lambda)$. This reading θ is the true rotation for the substance under investigation for some intermediate wave length. This wave length is found by eliminating θ between the equations

$$\frac{dT}{d\theta} = 0 \text{ and } \frac{dI}{d\lambda} = 0.$$

Hence it is evident that if $\theta = \theta(E, \rho)$ is a solution of the first of these, then the wave length λ_0 sought is given by $\lambda_0 = \theta(E, \lambda)$ or

$$\tan 2 \lambda_0 = \frac{\int E \sin 2 \lambda d \lambda}{\int E \cos 2 \lambda d \lambda}. \quad (4)$$

With any heterogeneous source then, a single setting of the analyzer may be made for minimum transmitted light. From this setting may be calculated one or more wave lengths for which this setting (increased or diminished by some integral multiple of π) is the actual rotation of the substance under investigation.

The minimum amount of light transmitted after the analyzer is set is given by (2) after the values of θ from equation (3) and λ from (4) have been substituted. This minimum value will evidently vary from a very small to quite a large quantity as the source is

less and less monochromatic. It is of importance in the discussion of errors in measurement later on.

In the following special cases the general equations above admit of complete solution and development in practical working formulas. They are taken up in turn.

(a) *Monochromatic Source*.—If the light used is so homogeneous that ρ may be regarded as a constant, (3) integrates into $E_0 \sin 2(\theta - \rho_0) = 0$, where E_0 is the total light transmitted by parallel nicols. In this case $\theta = \rho_0$ is a solution; the reading of the instrument gives the actual rotation directly, whatever the value of E_0 . Increasing the intensity of the source of light merely increases the accuracy of the setting. But spectrum lines always broaden with increase of intensity. Eventually the increased accuracy due to increased luminosity must be offset by a decrease in accuracy due to heterogeneity of the light used. For $\theta = \rho = \text{a constant}$, (2) gives $T = 0$, so that the minimum setting is complete extinction.

(b) *Two Monochromatic Sources*.—Let λ_1 and λ_2 be the wave lengths of the double source used (say the sodium lines), and let E_1 , E_2 , and ρ_1 , ρ_2 be the corresponding intensities and rotations. Then (3) gives as the condition for a minimum of transmitted light

$$E_1 \sin 2(\theta - \rho_1) + E_2 \sin 2(\theta - \rho_2) = 0 \quad (5)$$

Now θ must lie between ρ_1 and ρ_2 in value and hence $\theta - \rho$ can not exceed $\rho_1 - \rho_2$. Except in the measurement of rotations amounting to hundreds of degrees then, we are warranted in using the angle for the sine of the angle, hence (5) gives the working formula

$$\theta = \frac{E_1 \rho_1 + E_2 \rho_2}{E_1 + E_2} \equiv \frac{\rho_1 + K \rho_2}{1 + K}$$

where $K \equiv E_2 : E_1$, the ratio of the intensities of the two component sources. From this it appears that the intermediate wave length for which the reading of the analyzer is the actual rotation is given by

$$\lambda_0 = \frac{\lambda_1 + K \lambda_2}{1 + K}$$

neglecting the curvature of the rotation curve $\rho(\lambda)$ between λ_1 and λ_2 .

(c) *Any Symmetrical Source*.—When the components of a double

monochromatic source are equal in intensity, $E_1 = E_2$. The setting of the analyzer gives $\theta = \frac{1}{2}(\rho_1 + \rho_2)$, the correct reading for the rotation corresponding to the mean wave length $\lambda = \frac{1}{2}(\lambda_1 + \lambda_2)$. Since any symmetrical source may be considered as composed of pairs of monochromatic elements of equal intensity, it is evident that the analyzer will give directly the true rotation corresponding to the mean wave length until the source becomes so broad that the curvature of the rotary dispersion curve is no longer negligible within the range of wave lengths represented by the spectral width of the source.

(d) *Any Aggregate of Monochromatic Sources.*—For an indefinite number of monochromatic sources, equation (1) becomes

$$I = E_1 \sin^2 (\theta - \rho_1) + I_2 \sin^2 (\theta - \rho_2) + \dots$$

while the condition for a minimum (3) becomes

$$E_1 \sin^2 (\theta - \rho_1) + E_2 \sin^2 (\theta - \rho_2) + \dots = 0$$

hence the working formula is

$$\theta = \frac{E_1 \rho_1 + E_2 \rho_2 + E_3 \rho_3 + \dots}{E_1 + E_2 + E_3 + \dots} \quad (6)$$

determining the setting of the analyzer for which it gives the true

rotation for the wave length $\lambda = \frac{E_1 \lambda_1 + E_2 \lambda_2 + E_3 \lambda_3 + \dots}{E_1 + E_2 + E_3 + \dots}$.

2. HALF SHADE ANALYZER.

Suppose the analyzer to consist of two nicols, making a small angle $\alpha = 2\delta$ with each other. Let these two analyzing nicols make angles of θ_1 and θ_2 with the polarizer in its normal position. Then $\theta_2 - \theta_1 = \alpha$, the analyzing angle. Hence by equation (1), the light transmitted by the two halves of the analyzer is, putting the reading of the instrument $\theta = \frac{1}{2}(\theta_1 + \theta_2) = \theta_1 + \delta = \theta_2 - \delta$,

$$I_1 = I_0 \sin^2 (\theta_1 - \rho) = I_0 \sin^2 (\theta - \delta - \rho)$$

and

$$I_2 = I_0 \sin^2 (\theta_2 - \rho) = I_0 \sin^2 (\theta + \delta - \rho).$$

For a setting of the analyzer the two halves are equally illuminated, $T_1 = T_2$, or

$$\int E \sin^2 (\theta - \delta - \rho) d\lambda = \int E \sin^2 (\theta + \delta - \rho) d\lambda \quad (7)$$

which reduces (without approximation) to

$$\int E \sin^2 (\theta - \rho) d\lambda = 0,$$

an equation independent of the analyzing angle α and identically the same as equation (3) for a simple analyzer. Hence a half shade analyzer will give the same reading as a simple analyzer whatever the heterogeneity of the source. This reading

$$\theta = \frac{E_1 \rho_1 + E_2 \rho_2 + E_3 \rho_3 + \dots}{E_1 + E_2 + E_3 + \dots}$$

gives the rotation corresponding to the wave length

$$\lambda_\theta = \frac{E_1 \lambda_1 + E_2 \lambda_2 + E_3 \lambda_3 + \dots}{E_1 + E_2 + E_3 + \dots}$$

The minimum intensity of transmitted light is by (7)

$$\int E(\lambda) \sin^2 \delta d\lambda$$

which is never zero even for monochromatic light sources, and is larger the less homogeneous the source and the larger the analyzing angle 2δ used.

3. CONDITIONS FOR MAXIMUM SENSIBILITY AND ACCURACY.

Since an increase in the intensity of a nearly monochromatic source of light is usually accompanied by a decrease in its homogeneity while accurate polarimetric measurements demand a maximum of both intensity and homogeneity, the best compromise between the two can only be determined after a careful consideration of the conditions governing sensibility and accuracy. These conditions involve the photometric sensibility and threshold value

of human vision as well as the form of analyzer used and the form of the $E(\lambda)$ and $\rho(\lambda)$ curves.

With a simple analyzer consisting of a single nicol, any setting of the analyzer such that the total light transmitted is imperceptible is a reading of the instrument. Hence θ differs from ρ by not more than the amount that gives the integral in equation (2) a value equal to the least light perceptible of the color used. Hence in practice, instead of actual extinction we have

$$\int E \sin^2 (\theta - \rho) d\lambda < T_0$$

where T_0 is the least light perceptible of the color used. If the source is nearly monochromatic and of total intensity E , then

$$E \sin^2 \epsilon < T_0$$

where $\epsilon = \theta - \rho$ is the maximum error in a setting of the analyzer. Hence $\epsilon < (T_0/E)^{1/2}$. To halve the maximum error in a single setting then, it is necessary to increase the intensity of the source four times. Increasing the intensity of the source is not an effective means of increasing sensibility.

The threshold value T_0 varies enormously with different colors. It is about 0.0005 meter-candle³ in the blue-green, twice as great for the green of the mercury lamp, perhaps twenty times as great for sodium yellow and more than a thousand times as great for the hydrogen red. The reciprocal of T_0 may be regarded as a measure of the sensibility of the eye to light of a given color, and has been made the subject of investigation by Ebert,⁴ Langley,⁵ König,⁶ and Pflüger.⁷ I have taken the results of Ebert on two persons, König on two persons, Langley on four persons, and of Pflüger on eleven persons, representing over forty series of observations in all. This

³ This value 0.0005 meter-candle was supplied me by Dr. E. P. Hyde, of the photometry division of the Bureau of Standards, and is the result of much personal observation.

⁴ H. Ebert, *Wied. Ann.* **33**, 136; 1888.

⁵ S. P. Langley, *Phil. Mag.* **27**, 1; 1889.

⁶ A. König, *Beiträge Psy. Phys.*, Hamburg, 1891.

⁷ A. Pflüger, *Ann. Ph.*, **9**, 200; 1902.

data is so discordant that no mean curve of any value can be drawn, but the curve

$$V = e^{-a(\lambda - \lambda_m)^2}$$

with $a = 5$ and $\lambda_m = 5.1$ was found to be as good a mean as any that could be drawn, and it was adopted as representing the sensitiveness of a sort of mean standard eye. Calculated values are given below with values referred to the maximum at $\lambda = 510 \mu\mu$ as unity.

$\lambda = \begin{cases} 510 \\ 520 \\ 530 \\ 540 \\ 550 \\ 560 \\ 570 \\ 580 \\ 590 \\ 600 \end{cases} \mu\mu$	500	490	480	470	460	450	440	430	420
	520	530	540	550	560	570	580	590	600
$V =$	1.00	0.95	0.82	0.64	0.50	0.29	0.16	0.086	0.041
									0.017

The sensibility of a polarimeter having a simple analyzer will then vary by the square roots of these amounts with the wave length of light used, and will be fifty to one hundred times less in the red and violet than in the green, even with sources of the same intensity.

Since T_0 and E are expressible in the same units, a rough limit to the angle ϵ may be determined. For $E = 100$ meter-candles, a source appears of painful brightness to the eye. Giving to T_0 the mean value of 0.001 meter-candle, $\epsilon^2 = 10^{-3}$; $10^3 = 10^{-5}$ radian², $\epsilon = .003$ radian or about 0.17 degree.

As the source departs from homogeneity two additional sources of error appear; the limits of integration are so wide that $\rho(\lambda)$ can no longer be considered constant between them and the minimum transmission is not extinction. A setting on a minimum of intensity is much less accurate than a setting on extinction and decreases in accuracy as the value of the minimum increases. For an approximately homogeneous source the total energy transmitted is

$$E \sin^2(\theta - \rho)$$

where E is the integrated intensity of the source. Hence if $\delta\rho$ is the variation in the rotation ρ between the extreme wave lengths represented in the source $E(\lambda)$, and T_0 is the threshold value of the luminosity as above,

$$\delta\rho < (T_0: E)^{\frac{1}{2}}$$

If then a green line 0.1 $\mu\mu$ broad and having an intensity of 100 meter-candles is used, a rotary dispersion equal to that of 15 cm of

quartz will give an error less than the possible error in setting the analyzer (0.17 degree). $0.1 \mu\mu$ is about the width of the sodium lines as ordinarily produced in an oxy-hydrogen flame.

When discrete sources (such as the two D lines) are used, the minimum intensity of the light transmitted is $I = E (\delta\rho)^2$. In case sodium light is used in measuring the rotation of a plate 1 cm thick, $\delta\rho = 0.43$ degree $= 7.5 \times 10^{-3}$ radians, $(\delta\rho)^2 = .000056$, so that the minimum light transmitted is about .000056 of the whole.

Half-Shade Apparatus.—When a double analyzer is used, the accuracy of a setting depends primarily on the sensibility of the eye in judging of the equality of the illumination of adjacent fields. The eye is able to detect differences of about one-tenth of one per cent in intensity under the most favorable conditions. Aside from personal variations due to the observer and the amount of eye fatigue, etc., this sensibility varies to a slight extent with the color of the light used and with the absolute intensity of illumination of the halves of the analyzer field. The sensibility appears to be a maximum for a field intensity equal to that of diffuse daylight, or, say, 20 meter-candles, and falls off rather slowly for greater and for less intensity. It is a maximum for green or white light and falls off slightly toward the red and violet. But with intensities varying from 1 to 100 meter-candles and colors ranging from deep red to violet, the variation in visual sensibility is small in comparison with variations due to the individual observer and the amount of his fatigue.

Let σ be the fraction of the whole by which the illumination of two adjacent fields must differ in order that the difference may be just perceptible; 0.1 per cent to 5 per cent or more according to conditions. Then calling these two intensities T_1 and T_2 ,

$$T_1 = (1 + \sigma) T_2$$

But by equation (7)

$$T_1 = \int E \sin^2 (\theta - \rho + \delta) d\lambda$$

and

$$T_2 = \int E \sin^2 (\theta - \rho - \delta) d\lambda$$

Let ϵ be the error in a setting of the analyzer, that is, the amount by which the setting of the analyzer θ differs from the rotation ρ to be measured, or $\epsilon = \theta - \rho$. Hence for a narrow source

$$(\epsilon + \delta)^2 = (1 + \sigma)(\epsilon - \delta)^2$$

from which the error

$$\epsilon = \frac{1}{4}\sigma\delta,$$

or one-fourth the product of the analyzing angle and the photometric sensibility. In other words, the sensibility of the analyzer is inversely proportional to the analyzing angle over a wide range of intensity and color of the light used.

It is of interest to know what analyzing angle is best for maximum sensibility. When this condition is fulfilled, the derivative with respect to the analyzing angle, of the difference in the illumination of the two halves of the analyzer is zero, since $I_1 < (1 + \sigma) I_2$ and hence $I_1 - I_2 < \sigma I$, where σ is the photometric sensibility of the eye (about 0.1 per cent). Hence to determine the best analyzing angle

$$\frac{d}{d\delta}(I_1 - I_2) = \frac{d(\sigma I)}{d\delta} = \frac{d(\sigma I)}{dI} \frac{dI}{d\delta} = I \frac{d\sigma}{dI} + \sigma = 0 \quad (8)$$

or since $I = E\delta^2$, we have from (8) to determine the best analyzing angle ($a = 2\delta$)

$$\frac{\sigma}{E\delta^2} + \frac{d\sigma}{dI} = 0.$$

As to dimensions, it is to be noted that E and I are measured in the same units, say meter-candles, σ and δ are pure numbers. This condition shows that if the photometric sensibility σ were independent of the full illumination ($d\sigma : dI = 0$), the smaller the analyzing angle the greater would be the sensibility. But with an illumination below 1 meter-candle, we know that σ increases rapidly as I decreases. That is with fainter field illumination, the least difference that can be detected in the intensity of two adjacent fields becomes a larger percentage of the whole. The best analyzing angle to be used can not be fixed without more exact photometric

data on the values of $d\sigma : dI$. As a rough estimate consider $d\sigma : dI = -1$, a probable value when I is about 0.1 meter-candle and σ about two per cent. Then with a source $E = 500$ meter-candles, $\delta = (0.02 : 500)^{\frac{1}{2}} = 0.002$ radian = 6 minutes of arc. Hence a good value for 2δ , the analyzing angle, would be 0.2 degree.

Errors due to the spectral width of the source are the same when a half-shade analyzer is used as with a simple analyzer, the equation determining these errors being the same. The chief error in the use of a source of considerable spectral width is due to the change in the rotation to be measured within that width.

Comparing the sensibility ϵ_s of the half shade with that of the simple polarimeter ϵ_1 we have

$$\epsilon_1 = \left(\frac{T_1}{E}\right)^{\frac{1}{2}} \text{ and } \epsilon_s = \frac{1}{4} \sigma \delta = \frac{\sigma}{4} \left(\frac{T_2}{E}\right)^{\frac{1}{2}}$$

hence

$$\frac{\epsilon_s}{\epsilon_1} = \frac{\sigma}{4} \left(\frac{T_2}{T_1}\right)^{\frac{1}{2}}$$

where T_1 is the threshold value or the least perceptible illumination and T_2 is the mean full illumination in the half-shade analyzer. As a numerical illustration, take one per cent for the photometric sensibility of the eye ($\sigma = 0.01$), eighty minutes for the analyzing angle ($\sigma = 40' = 0.01$ radian). Then

$$\frac{\epsilon_s}{\epsilon_1} = \frac{0.01 \sqrt{0.01}}{4 \sqrt{0.001}} = 0.025$$

or the half-shade analyzer is forty times as sensitive as the simple analyzer.

The sensibility of either analyzer increases with the square root of the intensity of the source. As the errors due to a lack of homogeneity of the light used are negligible until the spectral width of the source is of the order of $0.5 \mu\mu$, the most intense sources at our command may be employed. *Unsymmetrical* sources, and particularly double unsymmetrical sources like the pair of D lines, are to be particularly avoided as leading to graver errors than the use of

symmetrical sources of much greater spectral width. Having determined upon a suitable intense source and knowing the relation between visual photometric sensibility and intensity of illumination, the best value of analyzing angle to give maximum polarimetric sensibility may be determined. The sensibility might then be still further increased by a method of photographic interpolation.

ON THE PLATINUM POINT ELECTROLYTIC DETECTOR FOR ELECTRICAL WAVES.

By L. W. Austin.

The electrolytic wave detector first described by Fessenden¹ and shortly afterward by Schlömilch,² consists essentially of a cell having as one electrode a fine point usually of platinum and a second larger electrode of platinum or some other metal. The cell in its most effective form contains an electrolyte, the decomposition products of which are gaseous. When an electromotive force is applied to such a cell, powerful polarization ensues, so that scarcely any current passes unless the electromotive force exceeds a certain critical value. When electrical oscillations pass through the cell, the resistance is decreased and the current is for the moment increased only to return to its former small value as soon as the waves cease. This phenomenon has been the subject of several investigations, but the conclusions reached by the several investigators³ have not been entirely in accord either in regard to the theory or the action of the instrument. According to Schlömilch and De Forrest it is highly sensitive to waves only when the point electrode is positive, and but very slightly so with negative polarization. Fessenden, and Rothmund and Lessing, on the other hand, claim that the cell is almost equally sensitive when the point is negative.

¹R. A. Fessenden, Barretter U. S. patent No. 727331. May 5, 1903, and *Electrotechn. Zs.* 24, p. 586 and 1015. 1903. A very similar instrument seems to have been used by M. I. Pupin for detecting small alternating currents. See *Bulletin Am. Phys. Soc.* 1, p. 21. 1900.

²W. Schlömilch, *Electrotechn. Zs.* 24, p. 959. 1903.

³M. Reich, *Physik. Zs.* 5, p. 338. 1904. L. DeForrest: Address before the Elec. Congress, St. Louis. 1904. V. Rothmund and A. Lessing, *Ann. d. Physik.*, 15, p. 193. 1904.

Fessenden holds that the decrease in resistance is largely a heat effect due to the energy of the waves, although other effects are mentioned in his patents, among them being the breaking down of polarization, and chemical action under the combined influence of waves and direct current, a phenomenon which has been already observed by Margules,⁴ Ruer,⁵ and others in the case of combined direct and alternating currents of lower frequency. Schlömilch advances no theoretical explanation, while Reich, and Rothmund and Lessing, ascribe the effect to a depolarizing influence of the waves, and the former also suggests chemical action. Rothmund and Lessing look upon the phenomenon in part at least as a type of rectification analogous to that of the aluminum rectifier. Schlömilch, and Rothmund and Lessing have also made an interesting study of the action of cells in which the electromotive force is supplied by the cell itself, the large electrode being of a metal different from the point. This is also the principle of the wave detector patented by Shoemaker.

The present work was taken up with the purpose, first, of studying the action of alternating currents of low frequency on the detector; second, the behavior of point electrodes of different sizes; third, the sensitiveness with different electrolytes; and finally, it was hoped that the results would explain the conflicting conclusions of the different experimenters and perhaps throw some light on the real nature of the phenomenon.

EXPERIMENTS WITH SLOW ALTERNATIONS.

It was thought that alternating currents of low frequency would give conditions which would be more reliable and more easily varied than would more rapid oscillations, and it was therefore determined to begin with currents of this character. The first cell constructed was filled with 30 per cent H_2SO_4 . The large electrode was of platinum foil having an area of about 2 sq. cm. The small point was of 0.2 mm platinum wire sealed into glass and broken off short. A point of this sort is not sensitive to any but very strong electrical waves but is highly so to slower oscillations. Afterward other

⁴ M. Margules, *Wied. Ann.* 65, p. 629, 1898; 66, p. 540, 1898.

⁵ M. Ruer, *Zeitschs. f. phys. chem.* 44, p. 81, 1903.

points, both smaller and larger, were used. The arrangement of the apparatus, which is shown in Fig. 1, was as follows:

The cell *B* was placed in series with a potentiometer *P* from which varying direct electromotive force could be applied, and a galvanometer *G*, which served to measure the current flowing through the cell. At (*a*) the circuit was broken and two wires led down to a fall of potential wire through which a known alternating current flowed. From this, by means of sliding contacts, any desired alternating electro-motive force could be impressed on the cell *B*.

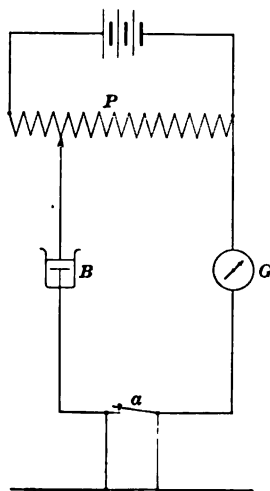


Fig. 1.

The amount of direct electromotive force applied to the cell was in general limited by the necessity of keeping the initial galvanometer deflection small, although in some cases this was accomplished by putting the galvanometer in shunt in a circuit which contained a second potentiometer.

TABLE I

H_2SO_4 (30 per cent.) See Fig. 2.

Polarizing E. M. F. 1.45 volts. Point 0.03 mm wire. Alternating current 60~.
Point Anode

Alt. E. M. F.	Increase in Direct Current in Cell
0.0003 volts	0.002×10^{-6} amp.
.0015	.010
.0030	.022
.009	.111
.012	.160
.015	.250
.024	.610
.027	.780
.030	.900

Table I shows the increase in the direct current due to various small alternating electromotive forces on a cell, the point electrodes

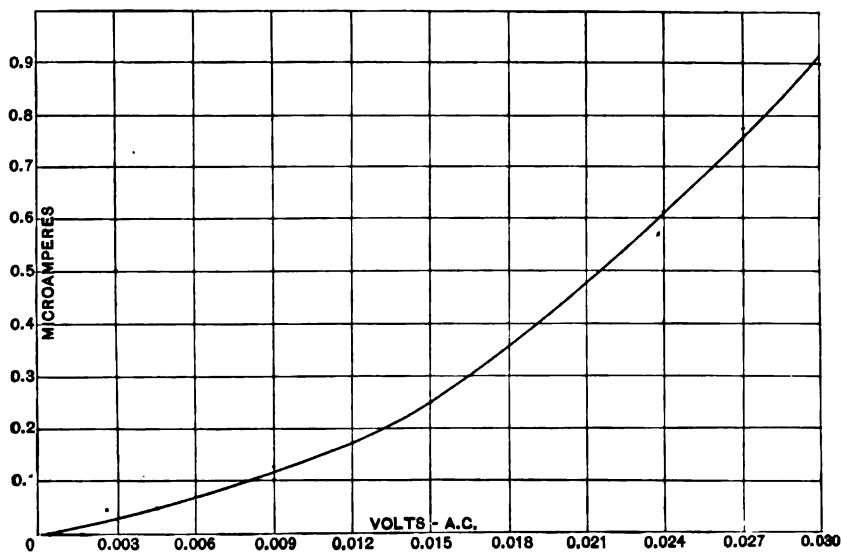


Fig. 2.

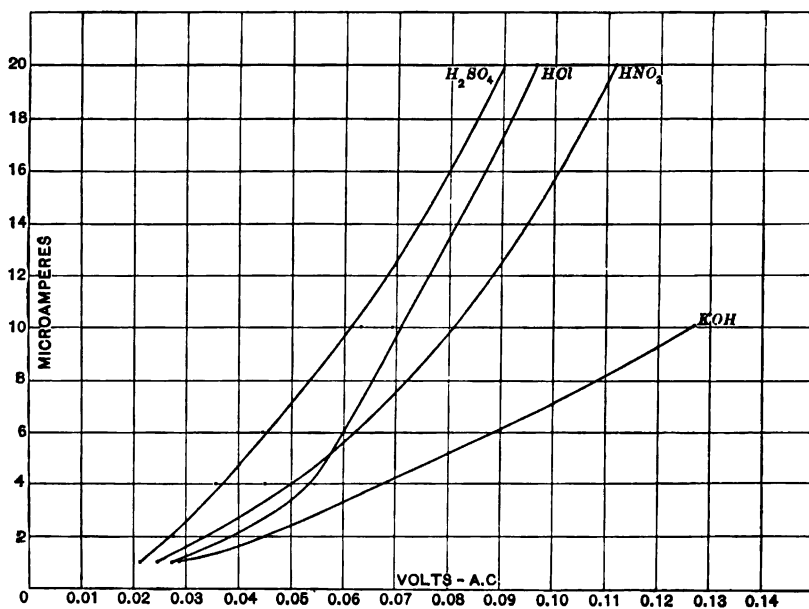


Fig. 3.

of which consisted of 0.03 mm wire broken off short in glass and polarized positively with 1.45 volts. Table II shows the sensitive-

TABLE II

Polarising E. M. F. 1.4 volts. Diameter of Pt point 0.3 mm. Alternating Current 60~. See Fig. 3.

HCl (50%) Point Anode Initial Direct Current $i=20 \times 10^{-6}$ amp. Alt. E. M. F. Increase in i		H ₂ SO ₄ (30%). Point Anode Initial Current $i=12 \times 10^{-6}$ amp. Alt. E. M. F. Increase in i	
0.027	1×10^{-6} amp.	0.021	1×10^{-6} amp.
.039	2	.027	2
.054	4	.035	4
.060	6	.045	6
.069	10	.063	10
.096	20	.090	20

Point Cathode Initial Direct Current $i=40 \times 10^{-6}$ amp.		KOH (50%) Point Anode Initial Current $i=10 \times 10^{-6}$ amp. Alt. E. M. F. Increase in i	
0.019	1×10^{-6}	0.027	1×10^{-6} amp.
.030	2	.045	2
.039	4	.066	4
.048	6	.090	6
.063	10	.126	10
.090	20		

HNO ₃ (20%) Point Anode Initial Current $i=60 \times 10^{-6}$ amp. Alt. E. M. F. Increase in i		HNO ₃ (20%) (Point 0.03 mm diam.) Point Anode Initial Current $i=10 \times 10^{-6}$ amp. Alt. E. M. F. Increase in i	
0.024	1×10^{-6} amp.	0.027	1×10^{-6}
.033	2	.039	2
.045	4	.051	6
.060	6	.087	10
.081	10	.135	20
.111	20		

Point Cathode Initial Current $i=50 \times 10^{-6}$ amp. Alt. E. M. F. Increase in i	
0.030	1×10^{-6}
.042	2
.057	4
.069	6
.087	10
.169	20

ness of different electrolytes to higher alternating electromotive forces. The polarizing electromotive force was in each of these cases sufficient to give maximum sensitiveness. This table shows the very approximate equality of sensitiveness of positively and negatively polarized points for alternating currents of low frequency. It is also seen that the 0.2 mm points are not less sensitive than the 0.03 mm.⁶ In Table III is shown the relation between sensitive-

TABLE III

HNO₃ (20 per cent)
Diameter of Pt. point 0.2 mm. Alt. E. M. F. 0.1 volt
Point Anode

Polarizing E. M. F.	Direct Current	
	Initial	Increase with Alt. Current
0.77 volts.	0.5×10^{-6} amp.	0.5×10^{-6} amp.
0.91	2.0	3.0
1.02	6	9.0
1.07	5	10
1.09	7	18
1.12	12	20
1.17	11	22
1.30	50	22
1.38	75	20

TABLE IV

H₂SO₄ (30 per cent)
Pt. point 0.2 mm. Alt. E. M. F. 0.1 volt.
Point Cathode.

Polarizing E. M. F.	Direct Current	
	Initial	Increase with Alt. Current
0.26 volts.	0.8×10^{-6} amp.	$+0.55 \times 10^{-6}$ amp.
.52	1.2	+ .55
.78	2.5	— .20
1.04	3.5	— .40
1.17	4.2	— .20
1.30	—	+ .80

ness and polarizing electromotive force. At about the point when the direct current begins to increase, the sensitiveness rises suddenly and remains constant or decreases slightly. The position of this

⁶ When the 0.2 mm wire was not broken off at the surface of the glass into which it was sealed, but was allowed to protrude about 4 mm, the sensitiveness was decreased.

point of rise in sensitiveness does not always correspond to the same polarizing electromotive force, but changes according to the conditions of the surface of the point in a way which is not yet understood.

Perhaps the most peculiar phenomenon noticed in the work is shown in Table IV. Here it is seen that with negative polarization on the point for small direct electromotive force on the cell, the direct current is diminished instead of increased when the alternating electromotive force is applied. This effect appears only occasionally and is less regular than the more common increase in current. It sometimes extends to much higher polarizations, and frequently there seems to be a combination of the two phenomena resulting in almost entire insensibility.⁷

EXPERIMENTS WITH ELECTRICAL WAVES.

In order to study the action of the electrolytic wave detector in the sphere in which it has the greatest practical interest, experiments were undertaken using electrical waves as the source of excitation. It was thought best to work only with waves of feeble intensity, partly because these are the only variety which are of interest in wireless work, and partly because preliminary experiments indicated that with powerful waves the action of the detector showed anomalies which it was not desired at this time to investigate. One source of waves used in the experiments was a small coil only 10 cm long run by two dry cells and capable of giving a spark perhaps 1 mm long. This was provided with vertical wires above and below, each about 50 cm in length serving as a kind of aerial. This sending apparatus was set up about 20 feet from the receiver, which was not supplied with an aerial. The key in the primary circuit of the small coil was operated by means of a string by the observer at the receiver. Signals were also sent frequently from the station of the National Electric Signaling Company in Washington by courtesy of Professor Fessenden. The station is about 6 miles from the Bureau, but as the Bureau aerial is very low and our instruments were not in tune, the signals were about as

⁷ This seems to be unconnected with the fact, first noticed by Fessenden, that with high polarizing electromotive force oscillations sometimes increase the resistance, which is probably due to the formation of large bubbles and occurs both with positive and negative polarization.

loud as those received at the Washington station of the company from a corresponding station 130 miles away.

In this work the chief questions to be considered, were, first, the relation between the sensitiveness of the detector and the polarizing electromotive force; second, the action under positive and negative polarization; third, the sensitiveness using different electrolytes; fourth, the absolute sensitiveness in terms of current and electromotive force.

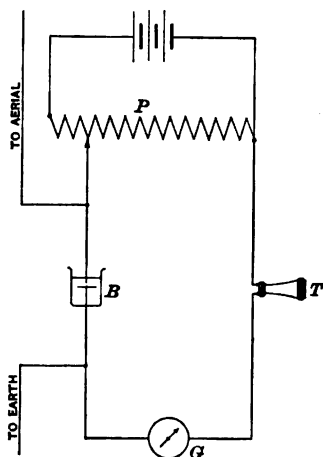


Fig. 4.

The arrangement of the apparatus is shown in Fig. 4. The detector *B* used in most of the experiments was of the type made by the National Electric Signaling Company, and consisted essentially of a minute platinum cup containing the electrolyte into which the tip of platinum wire about 0.002 mm in diameter

dipped. This platinum wire is drawn in silver, the silver being dissolved away in the acid, leaving the bare platinum point ready for use. A galvanometer *G* and a telephone *T* were in series with the detector and potentiometer *P*. The aerial and earth connections are shown in the figure.

POSITIVE POLARIZATION OF POINT.

Signals from Coil in Room.—Tables V and VI contain the results of the comparison of several electrolytes in the detector under various polarizing electromotive forces. HCl is of especial interest as in this case Cl and not O is deposited on the point. The much more decided breaking down of resistance as shown on the galvanometer suggests the idea of chemical action, the Cl attacking the Pt point more vigorously than O. Table VII shows a comparison of the sensibility for signals from the station of the National Electric Signaling Company. In this case the intensity was measured by observing with how small a shunt around the telephone the signals remained audible. In general at this distance the signals were too weak to produce any breaking down of resistance observable on the

TABLE V

HCl (50 per cent).
Point of 0.002 mm wire not in glass. Waves from small coil
Point Anode

Polarising E. M. F.	Direct Current	
	Without waves	With waves
1.3 volts.	0×10^{-6} amp	0×10^{-6} amp
1.6	0	3
1.8	0.5	4
2.0	0.0	8
2.2	0.5	25
2.4	1.0	40

TABLE VI

Point 0.002 mm wire not in glass. Waves from small coil.

	Polarising E. M. F.	Direct Current	
		Without waves	With waves
HCl 50%	2.4 volts	1×10^{-6} amp.	40×10^{-6} amp.
HNO ₃ 20%	1.6	5	8
H ₂ SO ₄ 30%	2.4	10	5
KOH 30%	2.6	5	15

TABLE VII

Point 0.002 mm wire not in glass. Waves from station of National Electric Signaling Company.

	Polarising E. M. F.	Shunt around telephone For silence
HCl 50%	2.0 volts	100 ohms
HNO ₃ 20%	2.3	250
H ₂ SO ₄ 30%	2.4	300
KOH 30%	2.7	300

Resistance of telephone, 1,300 ohms.

galvanometer. On two occasions, however, satisfactory galvanometer observations were also made. It is to be observed that the differences between the sensitiveness of the electrolytes are much smaller than for the stronger signals which affect the galvanometer. HCl is here also the most sensitive, but while the sensitiveness for distant signals as heard in the telephone is only about twice that of HNO₃, the galvanometer deflections were generally five to seven times greater.

NEGATIVE POLARIZATION OF POINT.

The question of the sensitiveness of a negatively polarized point is one on which there has been much difference of opinion. Schlömilch and De Forrest claim that there is almost no sensitiveness in this case, while Fessenden and Rothmund and Lessing claim a sensitiveness nearly equal to that with positive polarization. The first experiments with the coil in the laboratory seemed to support Schlömilch⁸ and De Forrest⁸ as no effect could be observed either on the galvanometer or in the telephone from the coil in the room except when the instruments were close together. But when the detector with negatively polarized point was connected to the aerial of the Bureau, signals from the station at the navy-yard and from the National Signaling Company's station were detected with about the same distinctness as with positive polarization. The cause of this difference in sensitiveness to signals from the small coil and to distant signals is extremely difficult to understand. First, it was thought that perhaps the waves from the coil were of too high frequency, but increasing the wave length by means of condensers failed to produce any effect. A second hypothesis was that the direct induction of the coil might produce an effect tending to counterbalance the influence of the waves. It has been already shown in Table IV that with negative polarization alternating currents sometimes produce a decrease of current instead of an increase, and it seems possible that we have here a combination of opposite effects which annul each other. It also seems possible that the difference in behavior was due to the fact that the wave train from the small coil was much more strongly damped than that from the distant station.

THE RESISTANCE OF THE DETECTOR.

For direct currents the resistance may be taken as the ratio of change in electromotive force to change in current. If the changes in electromotive force are small the numerical value of the change in current is the same whether the electromotive force be increased or decreased. When the polarization is nearly complete, i. e., when very little direct current flows, the resistance is very high; for

⁸ Loc. cit.

example, in the case of HCl (50 per cent) using a 0.002 mm platinum point polarized with 2 volts, an increase or decrease of 0.02 volt changed the current by 1.5×10^{-6} amperes; i. e., the resistance as above defined is about 13,000 ohms. As it did not seem certain that the resistance would be the same for alternating currents, an experiment was tried with a current of low frequency using the arrangement of apparatus shown in the figure. The resistance of the detector was determined by introducing in R the amount of resistance necessary to cut down by one-half the alternating current in the telephone, the current strength being measured relatively by noting the amount of resistance in

the shunt around the telephone necessary to produce silence. Neglecting the resistance of the (shunted) telephone and the potentiometer, R is equal to the resistance of the detector. In various trials this was found to vary between 9,000 and 15,000 ohms, which agrees with the value found above in the direct current experiments. When the electromotive force from the potentiometer is increased so that more direct current flows the resistance drops rapidly, in one experiment with HNO_3 (20 per cent) falling to about 300 ohms. When the polarizing electromotive force on the detector was zero the resistance was about 100,000 ohms. For high frequency oscillations it is probable that the electrolytic capacity of the point may play an important part in the effective resistance.

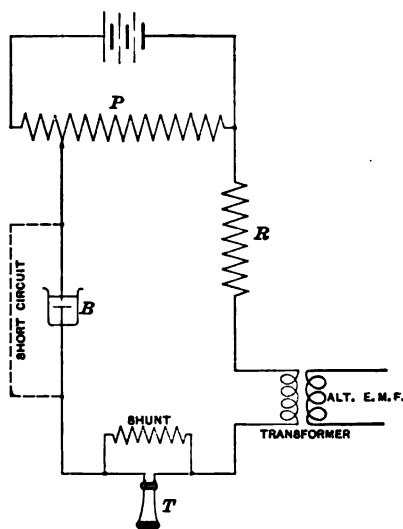


Fig. 5.

EFFECT OF HEATING THE POINT ELECTRODE.

According to Fessenden,⁹ the action of the detector depends to a considerable extent on the heating at the surface of the point due to the passage of the current through the large resistance which

⁹Loc. cit.

exists there. If this is the case, changes in the resistance between the wire and the liquid must be produced if some way of heating the point can be found. This was accomplished by constructing the small electrode in the form of a wire loop L , through which a heating current could be sent, and arranging the circuit as in Fig. 6.

Two sets of experiments were made with this form of apparatus, in the first of which 0.03 mm wire was used. When an alternating electromotive force of more than 0.01 volt was applied, the current through the direct circuit was increased, showing that the heating of the wire was sufficient to change the resistance of the cell. As no current could be detected in the galvanometer due to the heating when the polarizing electromotive force was zero, this was considered

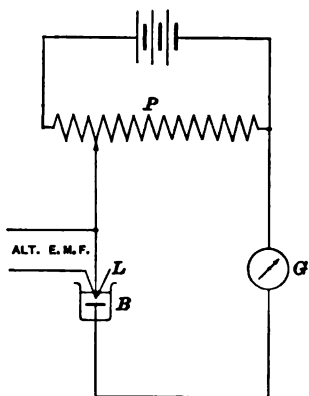


Fig. 6.

to prove that the effect was not due to thermoelectric action. This was also shown by the fact that the change in current was also an increase when the polarizing electromotive force was reversed. The second experiment was similar, except that in this case the loop was formed of 0.002 mm wire about 1 mm long, giving a resistance of some 50 ohms. This was so sensitive that when a telephone receiver was used as the source of alternating current in the loop, a loud shout was enough to increase the current in the galvanometer G by several micro-amperes.

As the resistance of the wire loop was small in comparison with that of the detector, it was not considered possible that enough of the alternating current could pass through the main circuit to produce a direct effect on the detector. This heating effect may be explained either by supposing that the change of resistance of the detector is due to a change of resistance in the electrolyte, or what seems more probable it may be due to a direct breaking up of the small polarization bubbles by the expansion of the gas. It indicates at any rate that the heating at the contact of the point electrode with the electrolyte is one of the factors of the phenomenon.

It seems probable that at least with strong signals there are several factors which have a part in producing the change in resistance

of the detector. The wearing away of the platinum point under the influence of oscillations, but not under direct current, a phenomenon familiar to every wireless operator using these cells, certainly points to chemical action. There is also probably a certain amount of rectification in the alternating current, for it is observed that when an electromotive force has been applied such that the polarization is at the point of breaking down, an increase in the electromotive force of more than about 0.02 volts increases the current more than a similar decrease in electromotive force decreases it. For rapid oscillations when the capacity action of the point electrode becomes prominent the rapid changes in electrostatic attraction across the gas film may tend to break up the minute bubbles mechanically. Estimates of the intensity of the electrical waves from the small coil indicate that the sensitiveness of the electrolytic detector for rapid oscillations does not differ materially from the sensitiveness for slowly alternating currents already determined.

CONCLUSIONS.

1. The electrolytic detector is sensitive to alternating currents of low frequency, and with a sensitive galvanometer is capable of detecting a few ten-thousandths of a volt. For the stronger alternating currents used, the breaking down in resistance is approximately proportional to the square of the alternating current. For the smaller currents it appears to vary with a lesser power.

2. With increasing polarizing electromotive force the sensitiveness increases to a maximum, after which it remains constant or decreases. This critical electromotive force is not constant, but varies according to the size and condition of the point electrode.

3. Under favorable conditions and with moderate polarization the detector is equally sensitive to alternating currents with the point electrode anode or cathode, but with the point cathode the action is oftener irregular, the resistance sometimes increasing instead of decreasing.

4. The resistance of the detector for slowly alternating currents varies from about 20,000 ohms to 400 ohms, according to the polarizing electromotive force applied. For rapid oscillations the capacity effect of the point electrode may play a part.

5. For electrical waves from a distance the detector is approximately equally sensitive with the point electrode anode or cathode, but for waves from a coil in the laboratory some cause appears to annul the sensitiveness of the cathode point electrode.

6. When the small electrode is made in the form of a loop of very fine wire and an alternating heating current is sent through it, the resistance of the cell is decreased, indicating that heat is one of the factors in the phenomenon. Probably chemical action, rectification, and electrostatic attraction across the gas film also have a part.

THE INFLUENCE OF FREQUENCY UPON THE SELF-INDUCTANCE OF COILS

By J. G. Coffin, Clark University, Worcester, Mass.

When currents of low frequency pass through the wires of a coil, the current distributes itself uniformly over the cross sections of the wires. With increasing frequency, this uniform current density no longer prevails, but, as is well known, at least for straight wires, the current density becomes greater at the surface of the wire at the expense of that of the interior.

The corresponding lines of magnetic force become differently distributed, and in consequence the self-inductance suffers a change. A short calculation will show the direction and amount of the change for circuits in which the curvature of the wire may be assumed negligible, and the theory derived for straight wires used.

The theory of this distribution of the current density in straight wires, which has been thoroughly worked out by Lord Rayleigh¹ and by Stefan,² is not applicable without modification to the distribution of current density in coils of wire.

The following argument shows that the effect of increasing frequency is to diminish self-inductance.

We shall assume, according to the theory, that with very high frequency the current flows entirely in the surface of the conductor. In computing the mutual inductance between two parallel circuits, we call their distance apart the logarithmic mean distance of the area of one cross section from the other. For circular areas, and circular lines, when completely outside one another, these distances are the same and equal to the distance between their centers. This

¹ Phil. Mag., 21, p. 381; 1886.

² Wied. Ann., 41, p. 400; 1890.

is also true for annular areas between any two concentric circles, that is to say, circular rings. When computing the self-inductance of a circle of circular cross section (solid tore), we use the value of the logarithmic mean distance $r=0.7788$ times the radius of cross section; but in computing the self-inductance of a circular tube (hollow tore), we must use for r the radius of the cross section.

To compute the self-inductance of n turns of wire, knowing the mutual inductances M_{mn} between each pair of turns, and the self-inductance L of each turn, we proceed by the following formula:

$$L_n = nL + 2(n-1)M_{12} + 2(n-2)M_{13} + \dots + 2M_{1n} \quad (1)$$

The deduction of this formula is an extension of the formula of the self-inductance of two equal coils in series:

$$L_2 = L + 2M_{12} + L$$

Hence, it is evident that however the frequency of oscillation may change, the mutual inductance between pairs of wires remains absolutely unchanged, while any change which does occur is due to that of the terms involving self-inductance only.

This difference is therefore n times the difference between the mutual inductance of two circular filaments when placed at a distance apart equal to $0.7788 \times$ radius of the cross section of the wire used, and when placed at a distance apart equal to the radius. For these are the factors used in computing the self-inductance of a turn for low and high frequencies, respectively. Thus, it is easily seen that since in the last case the self-inductance is less than in the first the influence of rapidly oscillating currents is to diminish the coefficient of self-inductance.

Calculations show that this difference is not negligible, so that in accurate work it is necessary to make a correction to the value calculated for low frequencies; this correction being a function of the frequency, the conductivity of the wire, the permeability of the materials of the coil, and of its configuration.

2. Lord Rayleigh has shown³ that for a straight conductor traversed by sinusoidal currents, the resistance and self-inductance

³ Loc. cit. See also Gray, *Absolute Measurements in Electricity and Magnetism*, Vol. 2, part I, p. 325 ff.

for any frequency are given by the following expressions in infinite series:

$$R' = R \left\{ 1 + \frac{1}{12} B^2 - \frac{1}{180} B^4 + \dots \right\}$$

and

$$L' = l \left\{ A + \mu \left(\frac{1}{48} B^2 + \frac{13}{8640} B^4 - \dots \right) \right\} \quad (2)$$

where $B = \frac{\mu \omega l}{R}$,

R is the resistance for steady currents,

μ is the magnetic permeability of the conductor,

l is the length of conductor,

ω is $2\pi \times$ frequency of the current.

A is determined from the following formula for the self-inductance for currents of low frequency:

$$L = l \left(A + \frac{1}{4} \mu \right),$$

$$R = \frac{l}{\pi \sigma \rho^2}, \text{ the resistance for steady currents,}$$

where ρ is the radius of the wire and σ is the specific conductivity of the wire.

When the frequency is great, or rather when the product of μ by the frequency is great, these expressions reduce to

$$R' = \sqrt{\frac{1}{2} \mu \omega R}$$

and

$$L' = l \left(A + \sqrt{\frac{\mu R}{2 \omega l}} \right) \quad (3)$$

For any frequency we can calculate the apparent resistance of the wire, and also the new value of the self-inductance, thus obtaining an estimate of the variation in self-inductance and resistance with any required frequency.

Calculations were made of this so-called skin effect with the following results:

TABLE I.

Effect of Frequency upon Resistance of Straight Wires.

A Frequency of current	B Thickness of Annulus Copper $\mu=1$	C Thickness of Annulus Iron $\mu=300$
80	0.719	0.0976
120	.587	.0798
160	.509	.0691
200	.455	.0617
480	.293	.0399
800	.228	.0309
1,000	.203	.0276
1,800	.152	.0206
3,200	.114	.0154
4,000	.102	.0138
9,000	.068	.0092
16,000	.051	.0069
20,000	.045	.0062
36,000	.034	.0046
40,000	.032	.0044
64,000	.024	.0035

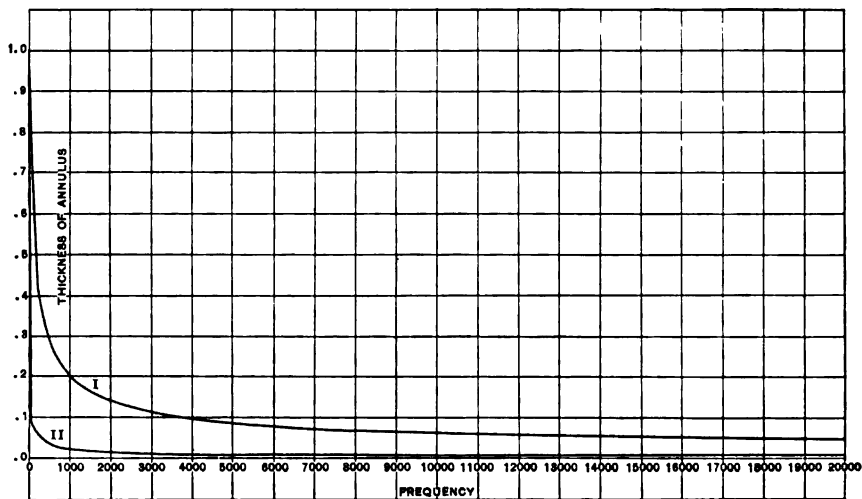


Fig. 1.—Curves showing the relation between skin-effect and frequency of alternation. Curve I for copper, $\mu=1$; curve II for iron, $\mu=300$.

The interpretation of Table I is as follows: As the frequency increases so does the resistance. Take a solid wire of any radius, and for a given frequency construct a hollow wire with the same external diameter, which would have the same resistance for steady currents as the apparent resistance of the solid one for alternating currents. Columns B and C give the ratios of the thicknesses of such hollow cylinders to the radius, for copper and iron, respectively.

Fig. 1 gives a graphical representation of these values. The abscissas represent the frequencies. It is seen how readily the current betakes itself to the surface of the conductor, so to speak. It shows that when the material is magnetic, a frequency of 1,000 gives a value of the inductance not appreciably different from frequencies of many times that amount. For copper this corresponding point is not reached before frequencies of 4,000 to 10,000, in the case for which these calculations were made.

3. The value of the self inductance for low frequency currents for a length of wire l is

$$L_o' = l (A + \frac{1}{2} \mu) \quad (4)$$

and the value for very high frequencies is shown by eq. (3) to be

$$L_\infty = lA = 2l \left(\log \frac{2l}{\rho} - 1 \right) \quad (5)$$

so that the difference is $\frac{1}{2} l \mu$; this difference is therefore directly proportional to μ . The coefficient of self-inductance of a straight conductor of circular cross section is given by

$$L = 2l \left\{ \log \frac{2l}{\rho} - (1 + \log 0.7788) \right\}$$

where ρ is the radius of the cross section of the conductor; which reduces to

$$L = 2l \left\{ \log \frac{2l}{\rho} - \frac{3}{4} \right\} \quad (6)$$

In the theory of the logarithmic mean distance, it is proved that the coefficient of self-inductance of any straight conductor of any cross section is equal to the coefficient of mutual inductance of two parallel straight filaments, at a distance apart equal to the

logarithmic mean distance of the cross section of this conductor from itself. For coils, the assumption is made that when the dimensions of the conductor are small in comparison with the radius of the coil, this law still holds.

Max Wien⁴ has shown that this assumption is very close to the truth, by the following example:

By integration of Maxwell's series for the mutual inductance between two parallel circles, over a circular cross section, that is by integration of

$$M = 4\pi a \log \frac{8a}{r} \left(1 + \frac{x}{2a} + \frac{x^2 + 3y^2}{16a^2} - \frac{x^3 + 3xy^2}{32a^3} + \dots \right) \\ + 4\pi a \left(-2 - \frac{x}{2a} + \frac{3x^2 - y^2}{16a^2} - \frac{x^3 - 6xy^2}{48a^3} + \dots \right) \quad (7)$$

where the radii of the two circles are a and $a+x$, and where y is the distance between their planes, and r the shortest distance between them, he obtains the following formula for the self-inductance of a circular coil of circular cross section:

$$L = 4\pi a \left\{ \left(1 + \frac{\rho^2}{8a^2} \right) \log \frac{8a}{\rho} - 1.75 - 0.0083 \frac{\rho^2}{a^2} \right\} \quad (8)$$

where ρ is the radius of the cross section of the wires and a is the mean radius of the coil. By employing the same number of terms of the series (7), and putting for y in the series 0.7788 times r he obtains, without integration, the following formula:

$$L = 4\pi a \left\{ \left(1 + \frac{0.91\rho^2}{8a^2} \right) \log \frac{8a}{\rho} - 1.75 - 0.0095 \frac{\rho^2}{a^2} \right\} \quad (9)$$

Assuming, now, such dimensions that ρ , the radius of the circular cross section of the wires is equal to one-fourth the mean radius a , the difference between the two values of L , computed by these formulas, is less than one part in a thousand. This is a severe test, as, in the application of the theory of the logarithmic mean distance, the dimensions of the cross section are assumed to be small in comparison with the mean radius. So that, in the use of this principle,

⁴Ann. d. Phys. und Chem., 53, p. 928; 1894.

we need not have the slightest fear of inaccuracy if the dimensions satisfy the above condition.

For example, in the application of the principle to the standard coil at Clark University, the difference between the two formulas is given by

$$4\pi a \{.00000437\}$$

which is negligible, producing a difference only in the eighth place of significant figures.

4. Using the first terms of Maxwell's series, i. e., neglecting squares of the ratio $\frac{y}{a}$ we obtain for the mutual inductance of two circles of the same radius at a distance y apart, the expression

$$M = 4\pi a \left\{ \log \frac{8a}{y} - 2 \right\}$$

Consider all coils to be wound in the form of a circle, and let the adjective applied refer to the form of the cross section. Then, for a solid circular coil, using the principle of the logarithmic mean distance

$$L = 4\pi a \left\{ \log \frac{8a}{r} - 1.75 \right\}$$

and for a circular tube

$$L = 4\pi a \left\{ \log \frac{8a}{r} - 2 \right\}$$

Their difference is πa , so that the *self-inductance of a single turn of wire and approximately of any coil of wire, diminishes by an amount numerically equal to half the length of the wire as the frequency becomes infinitely great.* The following computation shows the amount of this decrease for the Clark University coil:

Putting $a = 27.09$ cm

$$2r = .0584$$
 cm

the difference πa in the self-inductance of one turn is 85 cm in 2,434 cm or about 3.5 per cent of the value for steady currents.

Now, in any coil the mutual inductances between turns are independent of the distribution of the currents in the separate wires, as

long as this distribution is symmetrical with respect to the axes of the wires; and that is here assumed. The total self-inductance of the coil in question is about 0.2162 times 10^9 cm, and there are 716 turns. Hence, the total change for infinite frequency will be 716 times 85, or 60860 cm, which is about 0.286 per cent of the total value. This certainly is not negligible. It is necessary, then, to look more closely into this source of error.

Lord Rayleigh⁵ and J. Stefan⁶ have worked out the theory of this effect for straight wires, and have deduced formulæ showing the increase in resistance and the decrease in self-inductance with frequency. Later observers have assumed the effect to be the same in coils as for straight wires. This is not legitimate, as is clearly shown by Max Wien.⁷ In his paper he shows that the tendency of the current is to concentrate itself upon the inside surface of the coil, and he derives two formulæ for the increase of resistance and the decrease in self-inductance in circular coils of single and also of many layers, wound with wires of circular cross section.

These formulæ, however, come out as series which converge only for small values of the frequency. The formula for the decrease in self-inductance has but one term derived, and the calculation of the others seems to be very laborious. It is evident from this paper, then, that the maximum change in self-inductance, for a coil of a single layer, is equal to that which would be produced by decreasing the radius of the coil by an amount equal to the radius of the wire used in winding the coil.

Taking the numerical values for the mean radius and radius of the wire used above, and multiplying the coefficient $\frac{\partial L}{\partial a} = 0.00491$ by $da = 0.0602$ cm, we obtain 0.0003 henry as the total possible change in an inductance of 0.088 henry. This is about 0.34 per

⁵ Loc. cit.

⁶ Ann. d. Phys., 14, p. 1; 1904.

⁷ The coefficient $\frac{\partial L}{\partial a}$ was derived by direct calculation for the standard coil constructed for Clark University, by the writer.

The self-inductance was computed for a mean radius a , and then for a mean radius $a + da$; the difference gave of course the value of dL for a radius a ; dividing by da , the rate of change of L with respect to a was obtained. See Bulletin of the Bureau of Standards, Vol. 2, No. 1.

cent of the total amount and is certainly not negligible. This number compares very favorably with 0.29 per cent derived by an entirely different process.

The formulæ derived by Wien for a coil of a single layer are as follows:

$$R' = R \{ 1 + 0.272 \omega^2 \phi^2 - \dots \}$$

and

$$L' = R \left\{ \phi \left(\frac{r}{\rho} - \frac{256}{45\pi} \right) - \omega^2 \phi^3 \Gamma + \dots \right\} \quad (10)$$

where

R' is the resistance at frequency $\frac{\omega}{2\pi}$,

R is the resistance at frequency 0,

L' is the self-inductance at frequency $\frac{\omega}{2\pi}$,

c is the length of coil,

ρ is the radius of wire,

σ is the specific resistance,

m is the total number of turns,

r is the radius of coil,

Γ is a small constant,

ω is $2\pi \times$ frequency,

ϕ equals $\frac{2\pi^2 \rho^2 m}{\sigma c}$.

The first important result of this calculation is that the increase of resistance is thirty-two times as large for coiled wire as it is for the same length of straight wire. This shows the danger of using without modification, for coils of wire, the results of Lord Rayleigh for straight wires.

We should expect a corresponding difference for the change in self-inductance, but from his results this is not possible to verify, as he was principally interested in the change of resistance. Wien's calculations agree very well with the experiments of Dolezalek⁸ on the change of resistance for small frequencies.

The next successful attempt to find the change of resistance with frequency was made by A. Sommerfeld,⁹ who derived an expression

⁸ F. Dolezalek: Ann. d. Phys., **12**, p. 1142; 1903.

⁹ A. Sommerfeld: Ann. d. Phys., **15**, p. 673; 1904.

valid for all frequencies and which for small frequencies reduced to that of Max Wien. He also was principally interested in the change of resistance, so that the application of his method to the variation in self-inductance was left untried.

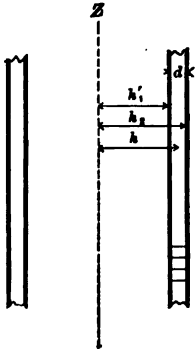


Fig. 2.

In the following will be found the derivation of a formula by a method based on that of Sommerfeld, valid for all frequencies, showing the change in self-inductance with the frequency. By means of this we shall be able to calculate the correction to be applied to any coil of a single layer.

5. *Derivation of the Formula.*—Consider an infinite circular current sheet of thickness d , inside radius r_1 , and outside radius r_2 .

Let the symbols $\mathbf{S} ()$ and $\mathbf{V} ()$ denote the scalar and vector products respectively. ∇ is the Hamiltonian operator. $\mathbf{V} \nabla ()$ is the same as curl $()$ or rot. $()$, while $\mathbf{S} \nabla ()$ is the same as div. $()$.

By symmetry, the field inside the coil is axial, and a function of r alone, at any given time, and on account of the coil being infinite in length, the field is all inside of it.

In the dielectric, Maxwell's equations are, if we neglect the displacement current,

$$\mathbf{V} \nabla H = 0 \text{ and } \mathbf{S} \nabla H = 0. \quad (11)$$

From the second relation we find that H is independent of z , and from the first relation, that it is independent of x and y ; hence it is constant inside the coil, and for

$$r < r_1, \text{ assume } H = H_0 e^{i\omega t} \quad (12)$$

where ω is $2\pi \times$ frequency, and H_0 the maximum value of H . Outside the coil, H must be constant, but as it vanishes at infinity it is zero everywhere. Hence, for

$$r > r_2, H = 0.$$

In the material of the coil, Maxwell's equations are

$$4\pi q = \nabla \nabla H, \frac{dH}{dt} = -\nabla \nabla F \text{ and } q = \sigma F \quad (13)$$

from which follow

$$\mathbf{S} \nabla H = 0, \text{ and } \mathbf{S} \nabla q = 0. \quad (14)$$

where

q is the current density

F is the electric force

σ is the specific conductivity.

Eliminating q and F , by taking the curl of the first equation (13)

$$\begin{aligned} 4\pi \nabla \nabla q &= \nabla \nabla \nabla \nabla H = \nabla \mathbf{S} \nabla H - \nabla^2 H \\ &= -\nabla^2 H, \text{ by (14),} \end{aligned}$$

and from the second and third of (13)

$$\begin{aligned} \nabla \nabla q &= \sigma \nabla \nabla F = -\sigma \frac{dH}{dt} \\ \therefore 4\pi \sigma \frac{dH}{dt} &= \nabla^2 H \end{aligned} \quad (15)$$

Now assume H in the material of the coil to be given by a function of r alone (on account of symmetry), multiplied by $H_0 e^{i\omega t}$, or $H = H_0 U e^{i\omega t}$

$$\begin{aligned} \text{where } U &= 1 \text{ when } r = r_1 \\ U &= 0 \quad \text{" } r = r_2 \end{aligned}$$

The differential equation then which U must satisfy becomes by equation (15) in polar coordinates

$$\frac{d^2 U}{dr^2} + \frac{1}{r} \frac{dU}{dr} + k^2 U = 0 \quad (16)$$

where $k = -i 4\pi \sigma \omega$.

The solutions of this equation are the Bessel's functions of order zero, J_0 and K_0 , and

$$H = H_0 \{A J_0(kr) + B K_0(kr)\} e^{i\omega t} \quad (17)$$

where A and B are arbitrary constants.

Now, Sommerfeld¹⁰ has shown that the expression

$$H = \sqrt{\frac{r_1}{r}} \frac{e^{ik(r_2-r)} - e^{-ik(r_2-r)}}{e^{ik(r_2-r_1)} - e^{-ik(r_2-r_1)}} e^{i\omega t} \quad (18)$$

is a very good approximation formula for the case in which kr_1 and kr_2 are large.

Since

$$4\pi q = \nabla \nabla H = -\frac{dH}{dr} \quad (19)$$

we obtain for the total current I per unit of length the value

$$I = \frac{I}{N} \int_{r_1}^{r_2} q dr = -\frac{1}{4\pi N} \int_{r_1}^{r_2} \frac{dH}{dr} dr = \frac{1}{4\pi N} (H_{r_1} - H_{r_2})$$

$$\therefore I = \frac{H_0}{4\pi N} e^{i\omega t} \quad (20)$$

where N is the number of turns per unit length of the coil.

6. The electrokinetic energy of a system is given by

$$T = \frac{1}{2} L I^2 \quad (21)$$

where L is its self-inductance, and I is the current. In the case of alternating currents, the mean square values are meant, or

$$T_1 = \frac{1}{2} L I_1^2$$

Hence we may write

$$L = \frac{2T_1}{I_1^2} = 2 \frac{\frac{1}{\tau} \int_0^\tau T dt}{\frac{1}{\tau} \int_0^\tau I^2 dt} \quad (22)$$

where τ is the period.

But the mean square value of an harmonic function is one-half the square of its amplitude, and in working with imaginary expressions the square of the amplitude can always be obtained by multiplying any expression by its conjugate.

¹⁰ Loc. cit., p. 678.

Hence

$$I_1^2 = \frac{1}{\tau} \int_0^\tau I^2 dt = \frac{1}{\tau} I \bar{I} \quad (23)$$

where $\bar{I}$ denotes the conjugate to I .

Therefore

$$\bar{I}^2 = \frac{1}{\tau} I I = \frac{1}{\tau} \left(\frac{H_0}{4\pi N} \right)^2$$

and

$$L = 4 \left(\frac{4\pi N}{H_0} \right)^{\frac{1}{2}} \int_0^\tau T dt. \quad (24)$$

In general

$$T = \frac{1}{8\pi} \int \int \int H^2 dv \text{ if } \mu = 1 \text{ everywhere,} \quad (25)$$

which in this case becomes, for unit length of coil,

$$T = \frac{1}{8\pi} \int_0^\infty H^2 2\pi r dr$$

$$\text{or,} \quad T = \frac{1}{4} \int_0^{\tau_1} H^2 r dr + \frac{1}{4} \int_{\tau_1}^{\tau_2} H^2 r dr = T_1 + T_2 \quad (26)$$

where, in T_1 , H is $H_0 e^{i\omega t}$

and in T_2 , H is given by equation (18).

We may write

$$\frac{1}{\tau} \int_0^\tau T dt = \frac{1}{4} \int r dr \frac{1}{\tau} \int_0^\tau H^2 dt = \frac{1}{4} \int H \bar{H} r dr.$$

$$\text{For } T_1 \text{ this becomes} \quad \left(\frac{H_0 r_1}{4} \right)^2 \quad (27)$$

To obtain the integral T_2 first put in eq. (18) $h(r+i)$ for ik .

Then, calling $h(r_2 - r) = y$,

$$\text{and} \quad h(r_2 - r_1) = y_1$$

we obtain for H and its conjugate $\bar{H}$ the expressions:

$$H = \sqrt{\frac{r_1}{r}} \frac{e^{iy} e^y - e^{-iy} e^{-y}}{e^{iy_1} e^{y_1} - e^{-iy_1} e^{-y_1}} H_0 e^{i\omega t}$$

and

$$\bar{H} = \sqrt{\frac{r_1}{r}} \frac{e^{-iy} e^y - e^{iy} e^{-y}}{e^{-iy_1} e^{y_1} - e^{iy_1} e^{-y_1}} H_0 e^{-i\omega t} \quad (28)$$

A conjugate expression is obtained by putting $-i$ for i everywhere in the equation.

Their product becomes

$$H\bar{H} = H_0^2 \frac{r_1}{r} \frac{(e^{2y} + e^{-2y}) - (e^{2iy} + e^{-2iy})}{(e^{2y_1} + e^{-2y_1}) - (e^{2iy_1} + e^{-2iy_1})}$$

or

$$H\bar{H} = H_0^2 \frac{r_1}{r} \frac{\cosh 2y - \cos 2y}{\cosh 2y_1 - \cos 2y_1}.$$

Multiplying this expression by rdr and integrating from r_1 to r , gives

$$\left(\text{since } \int \cosh r dr = \sinh r \right)$$

putting

$$x = 2h(r - r_1) = 2hd,$$

$$T_1 = \frac{H_0^2 r_1 d \sinh x - \sin x}{8 x \cosh x - \cos x}.$$

We may now write

$$L_\omega = 4\pi^2 N^2 r_1^2 \left\{ 1 + \frac{2}{3} \frac{d}{r_1} \left(\frac{3 \sinh x - \sin x}{x \cosh x - \cos x} \right) \right\} \quad (29)$$

where

$$x = 2d\sqrt{2\pi\omega\sigma} = (4\pi d\sqrt{\sigma})\sqrt{n} \text{ where } \omega = 2\pi n.$$

Now, $4\pi^2 N^2 r_1^2$ is the self-inductance per unit length of a coil with a mean radius r_1 , and is therefore the self-inductance for infinite frequency. This is also shown by eq. (29) as the second term in parenthesis is zero when $x = \infty$ for $\frac{\sinh x}{\cosh x} = 1$ and $\sinh x$ and $\cosh x$ increase without limit as x increases without limit, as may be seen by their expansions:

$$\begin{aligned} \sin hx &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots; \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots; \\ \cos hx &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots; \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \end{aligned} \quad (30)$$

Equation (29) may then be written

$$\frac{L_\omega - L_\infty}{L_\infty} = \frac{2}{3} \frac{d}{r_1} \left(\frac{3 \sinh x - \sin x}{x \cosh x - \cos x} \right) = \frac{2}{3} \frac{d}{r_1} Q. \quad (31)$$

By this equation we see that the fractional change of self-inductance in any coil may be calculated if the properties of the function

$$Q = \frac{3 \sinh x - \sin x}{x \cosh x - \cos x} \quad (32)$$

be known for all values of x from zero to infinity.

A formula for small frequencies may be derived by expanding the hyperbolic and circular functions and retaining only low powers of the variable x . Q then becomes

$$Q = \frac{\frac{1}{3!} + \frac{x^2}{7!} + \frac{x^4}{11!} + \dots}{\frac{1}{2!} + \frac{x^2}{6!} + \frac{x^4}{10!} + \dots}$$

which, retaining only terms in x^4 is

$$Q = \frac{\frac{1}{2} + \frac{3x^4}{7!}}{\frac{1}{2} + \frac{6!}{6!}} = \frac{1 + \frac{6x^4}{7!}}{1 + \frac{2x^4}{6!}} = 1 + \frac{6x^4}{7!} - \frac{14x^4}{7!} = 1 - \frac{8x^4}{7!} = 1 - \frac{x^4}{630}.$$

This equation is applicable for small frequencies only, and (31) may be written, since $64\pi^2 = 630$ very approximately,

$$\frac{L_\omega - L_\infty}{L_\infty} = \frac{2}{3} \frac{d}{r_1} \left(1 - \omega^2 d^2 \sigma^2 \right). \quad (33)$$

It agrees as to form and dimensions with that of Max Wien, eq. (10).

When $\omega = 0$, it reduces to $\frac{2}{3} \frac{d}{r_1}$. As a verification of the term $\frac{2}{3} \frac{d}{r_1}$,

the approximation formula for Bessel's formula with small argument

$$H = H_0 \sqrt{\frac{r_1}{r} \frac{r_2 - r}{r_2 - r_1}} e^{i\omega t} \quad (34)$$

was taken and a calculation similar to the one above was made for the general case. This completely verified this term, and proved that the expression used above, eq. (31), is valid for small arguments

as well as large. This last expression $\frac{2d}{3r_1}$ multiplied by $L_\infty = 4\pi^2 N^2 r_1^2$ is

$$L_0 - L_\infty = \frac{8}{3} \pi^2 N^2 r_1 d \quad (35)$$

and is therefore the additional self-inductance due to the field in the wires themselves for steady values of the current, or, in other words, this is the maximum possible change in self-inductance. Calculations by means of (35) show this expression to give, for coils of finite length and wires with round instead of square cross sections, the following values:

TABLE II.

Comparison of the Theoretical with the True Change in Self-Inductance.

Coil Length	True Change	Value of $\frac{2d}{3r_1}$	Ratio
46.0 cm.	.00038 henry	.00028 henry	1.35
30.5 "	.00020 "	.00016 "	1.25
15.0 "	.000067 "	.000057 "	1.17
13.0 "	.000051 "	.000044 "	1.16

The values in the second column were computed by taking the difference between the self-inductance of a coil of mean radius a , computed by means of an exact formula, and that of the same coil, using as a mean radius $a - \rho$, where ρ is the radius of the wire. These results, considering the assumptions made in the theory, are in good agreement with it. To make the theory fit the facts, the second term in the parenthesis in equation (29) should be multiplied by a constant, the average value of which is approximately 1.25, deduced from the above table. We should expect some multiplier to be necessary because a current sheet is the equivalent of square wires without insulation, while, in any actual coil, the wires are round and have insulation. We may then write

$$\frac{L_0 - L_\infty}{L_\infty} = 1.25 \frac{2d}{3r_1} \left(\frac{3}{x} \frac{\sinh x - \sin x}{\cosh x - \cos x} \right) \quad (36)$$

In the proposed use of this theory, however, it is not necessary to employ this constant, whose value at best is not satisfactorily determined.

As $\cosh x$ and $\sinh x$ increase without limit as x increases, equation (36) may be written, for large values of ω

$$\frac{L_{\omega} - L_{\infty}}{L_{\infty}} = 1.25 \frac{d}{r_1} \frac{3}{x} = \frac{5}{4} \frac{d}{r_1} \frac{1}{d\sqrt{2\pi\omega\sigma}} \quad (37)$$

This formula agrees to within a constant factor with that of Lord Rayleigh for large values of n , equation (3),

or
$$\frac{L_{\omega} - L_{\infty}}{L_{\infty}} = \sqrt{\frac{R}{2nl}} = 2 \frac{1}{d\sqrt{2\pi\omega\sigma}}.$$

Computations were carefully made for the values of the term

$$Q = \frac{3}{x} \frac{\sinh x - \sin x}{\cosh x - \cos x} \quad (38)$$

for the different values of x , as well as of Q_1 and Q_2 , where

$$Q_1 = 1 - \frac{x^4}{630}, \quad (39)$$

$$Q_2 = \frac{3}{x}. \quad (40)$$

Table III contains the results.

It is clearly seen that as long as $x < 2$ formula Q_1 may be used, while if $x > 7$, formula Q_2 is valid. Between these points the calculated values will serve.

A plot of these three curves shows still more clearly the relations of the three formulæ for different values of x . (See Fig. 3.)

7. *Manner of Using the Curve.*—This curve is to be used in the following manner. We assume that in any coil of a single layer the falling off of the self-inductance takes place according to equation (32), represented by the full line curve, and the maximum ordinate unity is taken to be the maximum possible change of the self-inductance, which may be calculated. For any given dimensions of the coil the

TABLE III.

Values for the Curves Q , Q_1 , and Q_2 for Values of x from 0 to ∞ .

x	$\cosh x$	$\sinh x$	$\cos x$	$\sin x$	Q	Q_1	Q_2
0.0	1.0000	0.0000	1.0000	0.0000	1.0000	1.0000	∞
0.5	1.1276	0.5211	0.8776	0.4789	0.9999	0.9999	6.0000
1.0	1.5431	1.1752	0.5402	0.8415	0.9984	0.9984	3.0000
1.5	2.3523	2.1291	0.0706	0.9975	0.9920	0.9918	2.0000
2.0	3.7622	3.6269	-0.4163	0.9092	0.9756	0.973	1.5000
3.0	10.0677	10.0178	-0.9900	0.1409	0.8932	0.871	1.0000
4.0	27.308	27.262	-0.6534	-0.7570	0.7515	0.594	0.7500
5.0	74.204	74.197	+0.2840	-0.9588	0.6100	Negative	0.6000
6.0	202.approx.	202.approx.	0.9603	-0.2790	0.5032	"	0.5000
7.0	573.approx.	573.approx.	0.7537	0.6574	0.4287	"	0.4289
8.0	Very large		-----	-----	0.3750	"	0.3750
9.0	and the		-----	-----	0.3333	"	0.3333
10.0	same as		-----	-----	0.3000	"	0.3000
	$\sinh x$						
∞	∞	∞	Indeter.	Indeter.	0	- ∞	0

values of x correspond to definite frequencies, and either a table or a plot of corrections may be derived from the curve.

For small values of x , and therefore for small values of the frequency, the correction varies proportionately to d^4 , while for very large frequencies it is proportionate to $\frac{1}{d}$ by equations (39) and (40).

As the first case is the more usual, it is important, in order to avoid the necessity for any correction, to wind the coil with wires of small diameter. The size necessary to avoid a correction under a given frequency may easily be calculated. As an example of the employment of this theory, the following example will be carried out:

Consider a coil of 27 cm radius = r_1 wound with a single layer of wire, whose diameter d is 0.063 cm. Let the conductivity of the wire be taken as 5.9×10^{-4} . Then the relation between x and n is, from (29) where n = frequency,

$$n = 2630x^2.$$

In the following table, column 2, are the values of x corresponding to the frequencies n in column 1. In column 3 are the values of the ordinates corresponding to these values of x taken from the curve, Fig. 3.

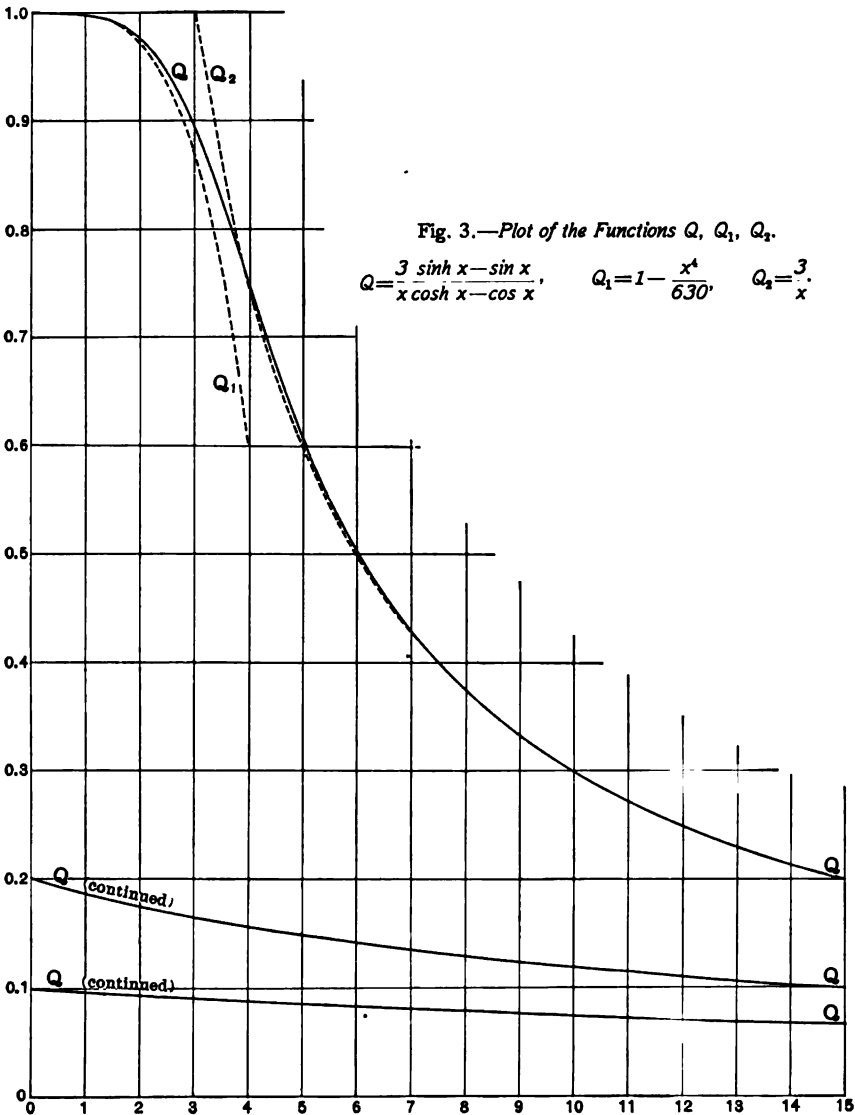


TABLE IV.

Values of Frequency n , and Corresponding Values of x and of Relative Change in Self-Inductance.

n	x	Ordinate
0	0	1.000
500	.44	1.000
1,000	.62	0.999
1,500	.76	0.998
3,000	1.07	0.996
4,000	1.23	0.995
5,000	1.38	0.993
6,000	1.51	0.990
10,000	1.95	0.977
15,000	2.38	0.950
20,000	2.75	0.916
30,000	3.38	0.846
40,000	3.90	0.765
50,000	4.35	0.700
60,000	4.78	0.638
70,000	5.15	0.590
80,000	5.51	0.550
90,000	5.85	0.515
100,000	6.17	0.483
120,000	6.76	0.463
150,000	7.55	0.397
170,000	8.05	0.373
200,000	8.72	0.344

The plot of the corresponding values of n as abscissas and values of column 3 as ordinates gives a curve of corrections for the above coil. For example, at a frequency of 50,000 the correction is 0.3 of the maximum change. The value of the correction may be obtained by reading *down* from the ordinate unity.

These computations also show that equation (39)

$$Q_1 = 1 - \frac{x^4}{630}$$

may be employed up to frequencies of 10,000 for this particular coil, and that equation (40)

$$Q_s = \frac{3}{x}$$

may be used for frequencies beyond 120,000 per second. As the maximum correction is only 0.2 per cent of the total, if frequencies under 4,000 per second are used, the error will not be over 0.2 per cent $\times 0.005 = 0.001$ per cent, which is practically negligible.

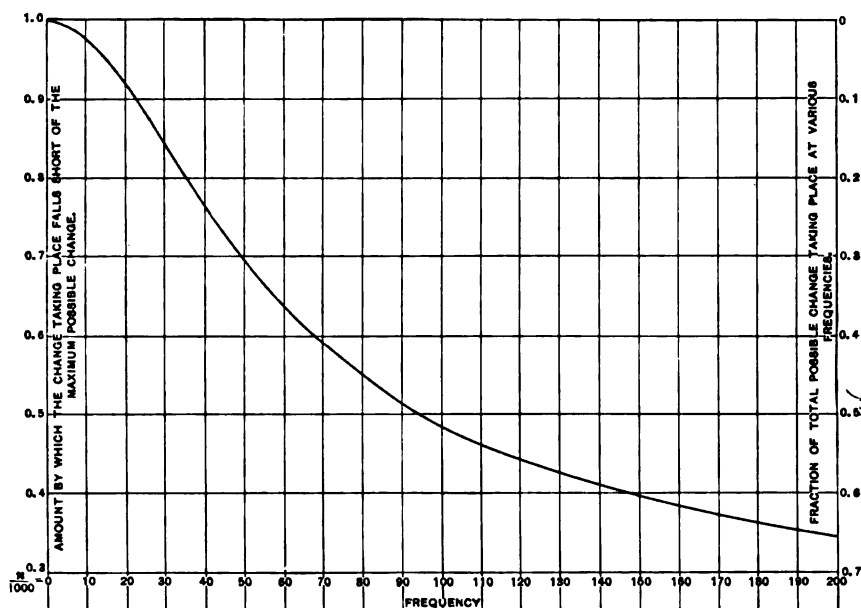


Fig. 4.—Curve showing the fractional part of the maximum possible decrease in inductance which takes place for different frequencies in a coil the wire of which has a diameter of 0.063 cm, the radius of the coil being 27 cm.

If, however, other things being equal, the diameter of the wire were four times larger than it is, the same corrections would correspond to frequencies 16 times less, or an error of 0.001 per cent is now obtained at a frequency of only 250.

This suggests that if large currents must be carried, to wind the coil with *flat* wires would diminish the effect of the frequency upon the changes both in self-inductance and in resistance.

Figs. 3 and 4 show how the self-inductance of a circular current sheet falls off with the frequency. In other words, the coil is assumed to be wound with wires of rectangular cross section. This may cause a doubt as to the applicability of the curve in question to the winding of round wires.

In the first place, the method of its use does away with the necessity of knowing the constant multiplier of the function Q , as explained above; and in the second place, the following two reasons go to show the entire correctness of the deductions.

Sommerfeld finds that his results agree to within a constant with those of Max Wien, which were deduced for circular wires, and both formulæ agree remarkably well with the experimental data at hand; this for the increase of resistance with the frequency. We should therefore fairly expect the results for the decrease in self-inductance also to be correct to within a constant factor. As there are no data available for the decrease in self-inductance, we have not been able to verify this conclusion. To further assure ourselves, however, we have worked out the problem of the change of self-inductance with the frequency in an infinite straight circular ring conductor, using the approximation formulæ for the Bessel's functions, as above given. It was found that the variable part of the expression for self-inductance agreed identically with that derived for a circular current sheet. This shows that the bending of a straight wire into a circular shape and forming a coil therewith does not affect the manner in which the self-inductance changes.

We believe, then, there is no doubt that Fig. 3 gives the law according to which the self-inductance falls off for straight ring-shaped conductors and probably for any simply shaped square or polygonal straight conductors, as well as for circular coils of a single layer, wound with wires of circular, square, or polygonal cross sections.

EXPERIMENTS ON THE HEUSLER MAGNETIC ALLOYS.

By K. E. Guthe and L. W. Austin.

Heusler's discovery¹ that it is possible to produce from so-called nonmagnetic metals alloys which approach iron in their magnetic properties has naturally aroused widespread interest. The discovery seems to have been accidental, Dr. Heusler's attention having been drawn to the phenomenon by the filings from some manganese alloys adhering to the tool with which he was working. His first observations were made on manganese-tin and manganese-copper-tin. While doubtless magnetic, they were but weakly so and until aluminium was tried as one of the components of the alloy no marked advance was made; the combination manganese-aluminium-copper, however, was found to be strongly ferromagnetic and the work was carried on for the most part with this alloy. The main results of the work by Heusler and his associates² were as follows:

(1) The magnetic properties of an alloy containing a given quantity of manganese are most pronounced when the ratio of aluminium to manganese is about one-half by weight—i. e., when one atom of manganese is present for each atom of aluminium. It was also found that the magnetizability increases more than proportionally as the relative amount of manganese and aluminium increases with respect to the other metal of the alloy. Unfortunately the alloys became at the same time extremely hard and brittle, so that with more than 28 per cent of manganese they are unworkable. Accord-

¹Fr. Heusler, W. Starck, and E. Haupt: *Verhandlungen der Physikalischen Gesellschaft*, 5, p. 219; 1903.

²Heusler, Starck, and Haupt: *Ueber die ferromagnetischen Eigenschaften von Legierungen unmagnetischer Metalle*, Marburg, 1904.

ing to the hypothesis of the discoverer, the manganese-aluminium-copper alloy is a solid solution of manganese-aluminium in copper.

(2) The alloys when first cast seem to have their molecules in a condition of unstable equilibrium and their magnetic properties can be much improved by heating them for many hours at a temperature of about 110° C. (The centigrade scale is used throughout this paper).

(3) At a certain temperature varying with the percentage composition between 70° and 300° the alloys lose their magnetizability, which generally returns when the temperature is again reduced. It was found in this connection that the presence of impurities, notably lead, reduced this critical temperature in a marked degree. Lead was also found useful in softening the alloys containing a large percentage of manganese, thus allowing the cast specimens to be turned in a lathe.

Tin, arsenic, antimony, and bismuth can be used instead of aluminium but with less favorable results. The highest value of induction observed was $B=6480$ for $H=150$ gauss in a specimen containing 24.1 per cent of manganese.

More recently the work on these alloys was taken up at the Reichsanstalt by E. Gumlich,³ who undertook a study of the permeability, coercive force, hysteresis loss and Steinmetz's coefficient under different temperature conditions. He found in agreement with the earlier observers that the permeability increased when the specimen was heated for many hours at 110° , but that heating for a long time at 165° lowered it again. The hysteresis loss was also greater at 165° than at lower temperatures. Maintaining the specimen at -190° for two hours appeared to have no influence on its magnetic properties. In one specimen examined the permeability attained a maximum (for $B=1100$) of 1200, i. e., equal to that of poor cast steel, but, as it fell to 35 for $B=3000$, this high maximum has more theoretical than practical interest. Steinmetz's coefficient was found to differ not materially from that observed in a poor cast steel. The maximum value of B observed was 4540 for $H=151$ in a specimen containing 23.5 per cent of manganese. An interesting feature observed is a large viscous magnetic after-effect.

³ *Annalen der Physik*, 16, p. 535; 1905.

One of the authors⁴ of the present paper, using the specimens belonging to Professor Gumlich, observed a magnetic expansion amounting to one-third of the maximum found in good soft iron. The form of the expansion curve is similar to that of magnetization, and a large after effect in the magnetostriction was noted.

Fleming and Hadfield⁵ have repeated the work of Heusler, Starck, and Haupt, using the alloys in the form of rings instead of bars. Their results agreed in general with those of the German experimenters, but the specimens used by them appear to have been of poorer magnetic quality.

The conclusions drawn by some experimenters, namely, that we have in these alloys entirely unmagnetic metals, and that magnetism is due to certain forms of molecular groupings, and is not an inherent characteristic of the substance—is not the only explanation of the phenomenon. It is true that the addition of manganese destroys the magnetic properties of iron, but it does not necessarily follow from this that manganese is in its nature nonmagnetic. According to G. Jaeger and St. Meyer⁶ the following suggestive series is formed by the molecular magnetism of nickel, cobalt, iron, and manganese:

Nickel	$I_m 10^6 = 2 \times 2.5$	c. g. s. units
Cobalt	"	$= 4 \times 2.5$ "
Iron	"	$= 5 \times 2.5$ "
Manganese	"	$= 6 \times 2.5$ "

According to Liebknecht and Wills⁷ the magnetic susceptibilities of certain salts are as follows:

$\text{Cu}(\text{NO}_3)_2 = 0.00163$	$\text{NiSO}_4 = 0.00435$
$\text{Ni}(\text{NO}_3)_2 = 0.00443$	$\text{Cr}_2(\text{SO}_4)_3 = 0.00599$
$\text{Cr}(\text{NO}_3)_3 = 0.00629$	$\text{CoSO}_4 = 0.01019$
$\text{Co}(\text{NO}_3)_2 = 0.01052$	$\text{FeSO}_4 = 0.01272$
$\text{Fe}(\text{NO}_3)_3 = 0.01352$	$\text{MnSO}_4 = 0.01514$
$\text{Mn}(\text{NO}_3)_2 = 0.01536$	$\text{Fe}_2(\text{SO}_4)_3 = 0.01515$

⁴ Austin: Verhandlungen der Deutschen Physikalischen Gesellschaft, 6, p. 211; 1904.

⁵ Proc. Roy. Society, 76, p. 271; 1905.

⁶ Wiener Berichte, 106, pp. 504, 623, 1897; 107, p. 5, 1898.

⁷ Annal. der Physik, 1, p. 178, 1900.

It appears, then, that manganese can have strong magnetic properties and belongs to the ferromagnetic group, but may, under certain conditions (those usually met with), be in a nonmagnetic state similar to the nonmagnetic modifications of iron. The resemblance between the two is a very close one, but, as already stated, the transformation to the unmagnetic state takes place in these alloys at a much lower temperature than in iron and seems to be more irregular. Take⁸ has shown that on repeated heating this transformation point is considerably raised, especially in alloys containing lead. In one specimen there was at first a slow change from 75° to 120° and on heating to 200° the critical temperature jumped suddenly to 235°. Heusler and Take believe that in some way on repeated heating the lead loses its property of lowering the transformation point. However, the sudden increase to a temperature much higher than that of a lead-free but otherwise chemically identical alloy (120° C.) can not be explained by this hypothesis. The most interesting result theoretically of Take's experiments seems to be that when the alloys are heated considerably higher than to what might be called the first critical point, namely to 520°, a number of the specimens became permanently nonmagnetic; this change being irreversible even at -185°. Two other specimens, however, acquired at 520° a higher magnetization, accompanied by a permanent increase in density, while two others were hardly affected by this high temperature.

These interesting results have been quite recently confirmed by Hill.⁹ He succeeded in transforming an alloy which was hardly altered at 500° into the nonmagnetic modification by still further heating to about 950°. The large decrease in density when the alloys become nonmagnetic was observed by Hill and interesting comparisons are drawn between the phenomena taking place in iron, nickel, and in Heusler's alloy.

The present experiments were undertaken to examine more fully the form of the magnetization curve of different specimens of the alloys, to obtain more data on the connection between the magnetization and magnetostriction which appeared to show such proportionality according to Austin's observations, and also to examine the

⁸ E. Take: *Verhandlungen der Deutschen Physik. Ges.*, 7, p. 133, 1905.

⁹ B. V. Hill: *Physical Review*, 21, 335, 1905.

relation between magnetostriction and thermoelectric force, since Bidwell's¹⁰ recent work on the close connection between magnetostriction of iron and nickel and their change of thermoelectric force in a magnetic field seemed to call for similar experiments on the Heusler alloys. Our investigation, in which we hoped to include a study of the influence of heat treatment on these properties, had unfortunately to be interrupted, due to one of the authors leaving the Bureau. While not as complete as we could wish, the work may nevertheless be of some interest.

The specimens investigated were obtained directly from Dr. Heusler, and are marked by the numbers 1 to 6; a specimen belonging to Austin and marked No. 0, was also included in our experiments. Dr. H. C. P. Weber, of the chemical division of the Bureau, has greatly added to the value of our data by making a very careful analysis of the first six specimens.

Chemical Analysis.

Number.....	1	2	3	4	5	6
Si	0.08%	0.07%	0.02%	0.16%	0.17%	0.05%
Pb	0.07	0.07	0.13	2.03	3.14	3.84
Cu	64.49	70.14	75.83	59.43	65.22	73.68
Mn	20.39	18.03	14.66	22.60	19.76	13.73
Al	13.25	10.03	8.64	14.50	11.13	8.33
Fe	1.05	0.99	0.55	1.31	0.67	0.46

In 1 and 2 no carbon could be detected, in 3 there was a trace of phosphorus, in 4 the silicon seemed to be combined chemically with either iron or aluminum, since when dissolved in hydrochloric acid the silicon was set free as a hydride. This reaction was characteristic for this specimen and did not occur with the others. As will be seen, No. 4 shows also a different magnetization curve from the rest, but our experiments were not carried far enough to decide whether this was due to the peculiar chemical constitution or to a heat treatment different from the others. Gumlich has shown how much a virgin piece may change on subsequent heating and it may be pos-

¹⁰ Proc. Roy. Society, 78, p. 413, 1904.

sible that No. 4 would have been improved magnetically by such treatment.

The seven specimens, which were carefully turned to a uniform diameter, were very hard and brittle, especially No. 1, while 4 did not differ much from 2; due to its high percentage of lead. No. 5 was still somewhat softer while 3 and 6 were not difficult to work. No heat treatment was given to any of the specimens.

In order to compare the results on magnetostriction with the modulus of elasticity, after the conclusion of our observations we asked the section of mechanical testing to determine that constant for us. Although this work was done very carefully on a Riehle testing machine, using a Johnson extensometer to measure the expansion, the pieces behaved in a very irregular way, most of them breaking under a rather small load and revealing in several cases flaws, the one in number 4 being as large as 50 per cent of the cross section. While it was apparent that Young's modulus is very large, we are unable to give even an approximate value for it.

The dimensions of the rods were as follows:

Number.....	0	1	2	3	4	5	6
Length, cm.....	17.20	13.60	12.13	12.00	14.80	12.45	12.65
Diameter, cm.....	0.60	0.83	0.87	0.87	0.87	0.88	0.87

MAGNETIZATION CURVES.

In order to be able to carry the magnetizing field to high values a powerful coil, consisting of 4,000 turns of No. 12 wire, of 51 cm length and 2 cm internal diameter was constructed. The field due to a current through this coil was determined at different distances from the ends by means of an exploring coil and was found to be practically constant over a length of about 20 cm in the center; for this distance the magnetizing field was given by the equation $H = 100 I$ gauss, if I is expressed in amperes. In order to avoid a possible heating effect in the magnetostriction experiments a double walled tube for the circulation of water was fitted inside the coil, but it was found unnecessary to make use of it, since the heating was appreciable only with currents above 5 amperes and even then

the time lag was large enough to complete the set before the heating became troublesome.

The ballistic method was employed for the magnetization tests. The secondary coils consisting of double silk-covered wires were wound in the case of alloy No. 0 directly on the rod, in all other cases on a thin brass tube fitting snugly over the bars. The secondary always covered the whole length of the test piece. The ballistic throw was measured by a galvanometer having a period of twelve seconds and calibrated by means of a standard mutual inductance which remained in the circuit throughout the test. The calibration was checked before every set of observations.

The total flux through the secondary is given by the formula:

$$\phi = (AH + 4\pi Ia) n \text{ lines}$$

where A is the cross section of the secondary,

a the cross section of the test piece,

H the actual magnetizing field,

I the average intensity of magnetization,

n the number of secondary turns.

In our case the field strength is greatly modified by the end effects, for which correction may be made by assuming for the actual field the formula:

$$H = H' - NI.$$

where H' is the original field as calculated from the current and N a quantity depending on the dimensions of the magnetized rods. Mann¹¹ has shown that up to medium magnetization (for iron up to $I = 400$) N is practically a constant; for higher values H is slightly incorrect but the general form of the curve is not changed and in our comparison of the magnetization curves with the magnetostriction the errors thus introduced are entirely negligible.

Substituting the value of H in the formula for ϕ we obtain

$$\phi = \{AH' + (4\pi a - AN)I\}n$$

$$\text{and } I = \frac{\phi n - AH'}{4\pi a - AN}$$

¹¹ C. R. Mann: Phys. Rev., 8, p. 359; 1896.

These formulæ were used for the calculation of I in the following, and B was obtained from the well-known relation between B , H , and I . N was taken from curves constructed on the basis of Mann's values.

The throw of the galvanometer needle was observed on the reversal of the current. It is well known in the case of iron, and has been shown by Gumlich to be true also for the alloys, that the curve will depend to a certain extent upon the steps chosen. Since we were concerned mainly with comparisons only, we tried to make equal steps as far as possible in the different cases. The results are given in the following tables and curves for B , I , and μ plotted in Figs. 1

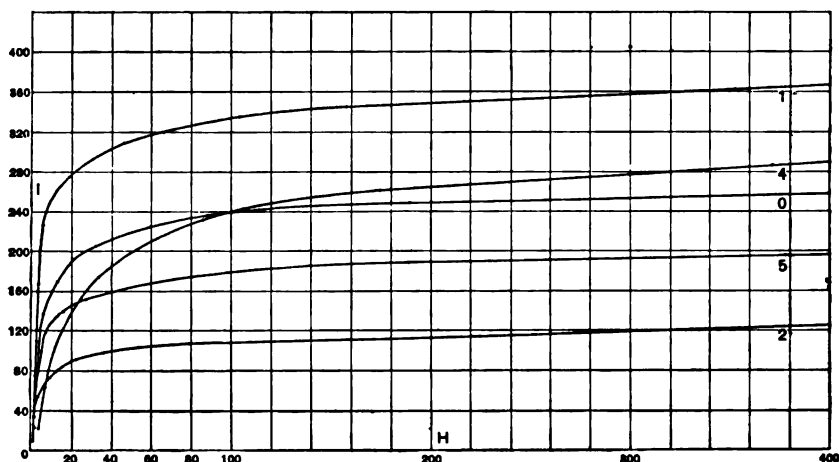


Fig. 1.—Curves for the Intensity of Magnetization.

to 4. The high induction in specimen 1 should be noticed as well as the peculiar curves for 4 showing the characteristic form of magnetically hard substances. No magnetic data are given for specimens 3 and 6. These showed even in the strongest fields such small magnetization that they could be considered as practically nonmagnetic and unsuited for the comparisons desired.

TABLE 1.

Alloy No. 0; N=0.045; n=548.

Current	H	I	B	μ
0.022 amp.	1.0	19.3	244	244
0.059	2.7	72.0	907	336
0.128	6.5	139.2	1756	270
0.155	8.6	153.0	1921	224
0.205	12.9	170.0	2150	167
0.403	31.2	204.2	2594	83
1.039	93.3	235.0	3043	33
2.00	189	247.5	3294	17
3.98	386	255.8	3600	9
6.28	615	259.7	3875	6

TABLE 2.

Alloy No. 1; N=0.121; n=440.

Current	H	I	B	μ
0.0445 amp.	0.8	23.6	298	372
0.12	1.8	78.3	986.5	544
0.16	2.4	109	1373	572
0.24	3.8	167	2102	553
0.351	7.9	225.5	2838	359
0.52	19.1	272	3437	180
0.859	48.5	309	3932	81
1.424	102.0	334	4300	41
2.00	157.9	348	4528	22
3.96	352.3	361	4897	13
5.78	534.0	364	5106	9
6.98	653.7	366	5254	7

The results show (as found by former observers) that of the lead free specimens 1 and 2 the former reaches a considerably higher magnetization, due to the larger percentage of the Manganese-aluminium compound, while 3 with only 14.66 per cent of manganese is nonmagnetic. Both 4 and 5, containing lead, lie in the main

TABLE 3.

Alloy No. 2; $N=0.155$; $n=394$.

Current	H	I	B	μ
0.022 amp.	0.7	9.25	117	167
0.0475	1.1	21.64	274	258
0.0779	1.8	36.95	441	220
0.0967	2.6	45.47	573	216
0.1282	3.9	57.5	726	187
0.133	4.1	59.5	752	183
0.207	8.9	76.2	966	109
0.354	21.5	89.8	1150	53
0.501	35.2	96.4	1245	35
0.940	78.2	105.5	1404	18
2.4	222.8	114.3	1661	8
4.49	421.1	119.5	1921	5

TABLE 4.

Alloy No. 4; $N=0.115$; $n=366$.

Current	H	I	B	μ
0.058 amp.	2.7	23.6	299.4	111
0.155	7.4	69.7	870	117
0.257	13.0	108.5	1374	106
0.500	30.22	172	2190	72
1.12	85.4	231	2985	35
2.04	174	260	3439	20
4.03	371	278	3863	10
6.23	590	284	4157	7
9.15	882	286	4474	5

between the other two, 4 showing for weak fields a smaller permeability than all the rest. The permeability of 0 is at first smaller than that of 2 or 5, though in maximum induction it exceeds either of them.

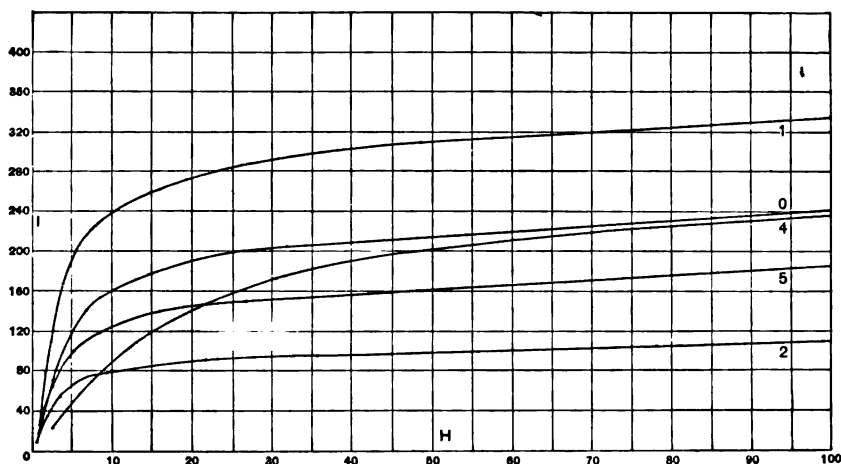


Fig. 2.—Curves for the Intensity of Magnetization.

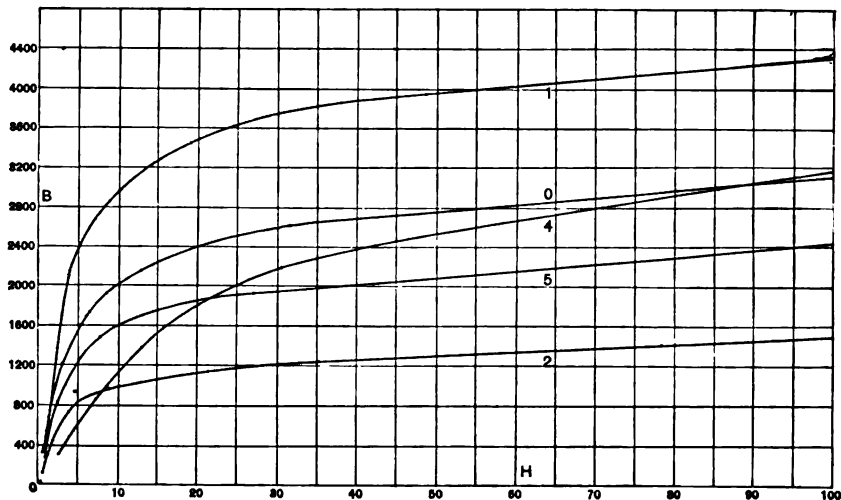


Fig. 3.—Curves of Magnetic Induction.

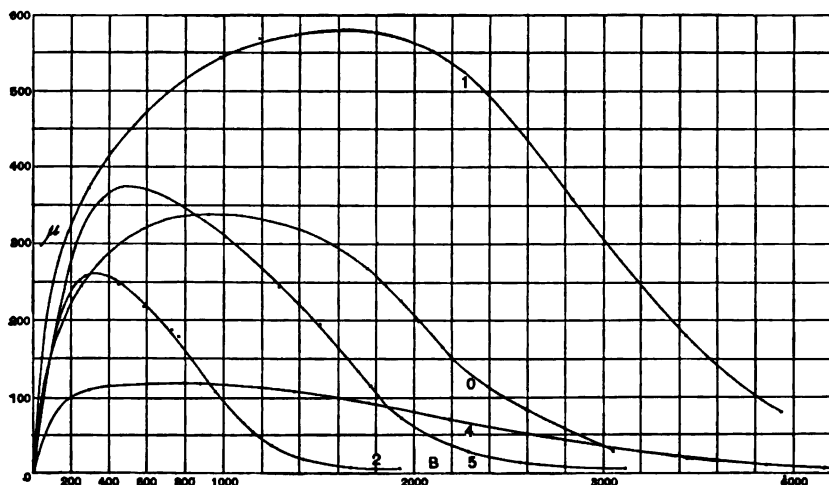


Fig. 4.—Curves of Permeability and Induction.

TABLE 5.

Alloy No. 5, $N=0.1515$, $n=400$.

Current	H	I	B	μ
0.058 amp.	1.0	28.4	358	358
0.128	2.5	67.1	845	338
0.207	5.25	101.7	1281	244
0.255	7.6	118.9	1499	197
0.368	15.6	140	1775	112
0.499	27	151.3	1928	71
1.06	80	175.1	2280	28
2.17	189.5	188	2550	13
4.05	375.4	195.0	2825	8
6.20	590	200.2	3108	6

MAGNETOSTRICTION.

The apparatus used in measuring the magnetostriction was constructed as follows: The magnetizing coil a , already described under magnetic measurements, was fixed horizontally on a wooden base (see Fig. 5). The specimen of alloy L was attached at both ends to brass tubes M , which were long enough to extend beyond the

ends of the coil. Behind the coil a wooden block was screwed to the base, to the top of which was fastened a brass clamp *c*, in which was inserted the end of one of the brass tube extensions of the bar. The other brass extension projected from the front of the coil and was in contact with the apparatus for measuring the change in length. The brass extension piece in front was supported and pressed back against the block in the rear by means of a slotted flat

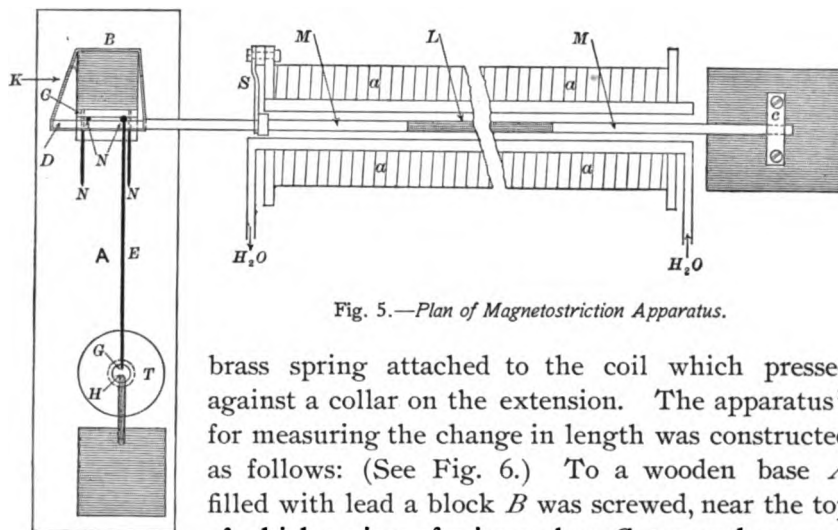


Fig. 5.—Plan of Magnetostriction Apparatus.

brass spring attached to the coil which pressed against a collar on the extension. The apparatus¹² for measuring the change in length was constructed as follows: (See Fig. 6.) To a wooden base *A* filled with lead a block *B* was screwed, near the top of which a piece of mirror glass *C*, 2.5 cm long and 2 cm wide, was cemented. Against this a second glass piece *D*, 3 cm long and 2 cm wide, was pressed by means of a rubber band *K* passing around the block. Between the two glass plates were placed two vertical sewing needles *N*, 0.34 mm in diameter, which rolled whenever the glass *D* was moved forward or backward. The inside surfaces of the two glass plates were slightly ground with the intention of giving a better hold on the needles. The bottom of the moving plate also rested on needles rolling on a plate below. To the top of one of the vertical needles a very light glass arm *E*, 10.1 cm long, was attached by a small brass cap. The end of the brass piece extending from the front of the specimen of alloy pressed against the movable glass plate; thus any expansion or contraction

¹² This was the form of apparatus devised by Austin for measuring the magnetostriction of the alloys in the work already cited.

caused the needle to roll and was indicated by the glass arm, magnified in the ratio of the length of the arm to the diameter of the needle. In order to further increase the magnification of the motion, we made use of Lord Kelvin's double suspension mirror method, i. e., a very light mirror F was attached to the end of the glass arm, as shown in Fig. 6. One end of a loop of cocoon fiber was fastened to the end of the arm at G , and the other end to a fixed point H , the mirror being attached to the loop by a paper hook. The distance between the points of suspension of the threads was always about 1.6 mm. The swings of the mirror were made nearly aperiodic by means of a small water damper.

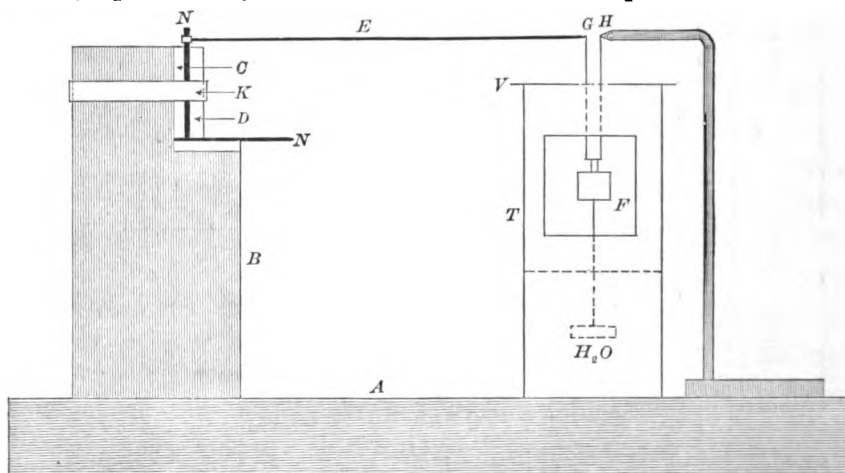


Fig. 6.—End View of Apparatus for Measuring Changes in Length.

The mirror and the threads were almost completely surrounded by a brass vessel T with a glass front for observing the deflections of the mirror. The top of the vessel was covered with a piece of cardboard containing a narrow slit just wide enough to allow a free movement of the threads. Thus the mirror was effectively protected from disturbing air currents.

The sewing needles were the only portions of the apparatus which were made of iron. Preliminary experiments showed, however, that these were not disturbed when the coil was excited. Experiments were also made to detect any movement of the magnetized body as a whole, but none was observed, if the rods as in our experiments, were placed in the center of the coil.

This apparatus is exceedingly well fitted for demonstration purposes, as with the exception of the coil it can be constructed in a rough but perfectly usable form in an hour. It is very little troubled by vibrations, if a water damper is used, and is simple in adjustment. By placing the points of suspension of the mirror nearer together it is quite possible to make the magnetic expansion of a 20 cm rod of iron produce a deflection of a spot of light on a screen 3 meters away, of more than a meter.

The principal constants of the expansion apparatus were as follows:

Length of glass arm $E = 101$ mm

Distance from mirror to scale $l = 1,135$ mm

Diameter of needle $\delta = 0.35$ mm

Distance between suspension threads $d =$ about 1.6 mm.

All of these quantities remained the same throughout the work with the exception of d , which was measured after each set of experiments by means of a filar micrometer, calibrated by comparison with a standard scale. The magnification ratio was:

$$R = \frac{2 El}{\delta d} = \frac{675000}{d}$$

If we assume d equal to 1.6 mm, and that 0.2 mm is the smallest scale deflection which can be read with certainty, the smallest displacement recognizable would be 5×10^{-7} mm. The deflections were very consistent and were repeatedly checked with the different specimens. The expansion followed the formation of the magnetizing field instantaneously, and no after effect, either in expansion or in contraction was observed, even with the strongest fields.

The results of the observations are shown in the following tables and curves.

TABLE VI.

Alloy No. 0; d=1.57 mm.

Current	H	I	Deflection	$\frac{dL}{L} 10^2$
0.055 amp.	2.5	70	2.0 mm	0.27
0.121	6.0	135	14.0	1.89
0.195	11.6	167	23.0	3.11
0.295	21.0	190	32.0	4.33
0.425	33.2	206	41.0	5.55
0.690	58.0	220	48.0	6.50
1.145	104.0	237	62.0	8.40
2.50	240	250	67.0	9.07
5.80	572	259	82.0	11.10
8.60	843	261	83.0	11.23

TABLE VII.

Alloy No. 1; d=1.63 mm.

Current	H	I	Deflection	$\frac{dL}{L} 10^2$
0.061 amp.	1.0	32	0.2 mm	0.04
0.163	2.6	116	5.8	1.03
0.340	7.3	220	18.3	3.23
0.522	19.3	273	38.8	6.90
1.03	64.0	316	61.8	10.95
1.65	124.0	338	75.5	13.45
2.51	209.0	348	85.5	15.20
6.15	568.0	365	97.3	17.35

As may be seen from the tables and curves, the expansion curves are quite similar in their general form to those for the intensity of magnetization with the difference, characteristic for all the alloys, that the expansion is very small for weak fields; it is, in fact, hardly appreciable for fields smaller than one gauss. The more pronounced the magnetic properties the larger the relative expansion, reaching

TABLE VIII.

Alloy No. 2; d=1.54 mm.

Current	H	I	Deflection	$\frac{dL}{L} 10^7$
0.057 amp.	1.4	25	0.5 mm	0.08
0.152	5.1	66	2.5	0.47
0.305	17.0	86	7.7	1.46
0.490	34.2	94	8.3	1.57
0.912	75.0	105	11.0	2.06
1.80	164.0	112	13.5	2.54
2.25	207.0	114	14.0	2.64
5.60	527.0	120	15.0	2.82

TABLE IX.

Alloy No. 4; d=1.57 mm.

Current	H	I	Deflection	$\frac{dL}{L} 10^7$
0.057 amp.	2.7	23	0.5 mm	0.08
0.150	7.0	65	6.3	0.96
0.300	15.7	122	17.0	2.70
0.485	29.3	169	30.0	4.84
0.895	65.0	220	52.5	8.25
1.25	98.0	238	65.0	10.22
1.60	130.0	248	71.0	11.15
2.20	188.0	262	73.8	11.63
5.40	510.0	282	82.5	12.98

in alloy No. 1 one-half the maximum found for soft iron. A contraction in stronger fields, as high as 1,000, has not been observed.

A direct comparison between magnetostriction and the intensity of magnetization is given by the curves of Fig. 8. It is seen that the expansion is not proportional to the magnetization, but increases more rapidly. It is interesting to note that the expansion is relatively larger the softer the material. As stated above, No. 1 is the hardest, 2 and 4 follow and are about equally hard, while 5 is relatively the softest. The exceptional position of 4 has in this

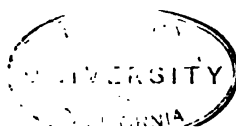


TABLE X.

Alloy No. 5; $d=1.63$ mm.

Current	H	I	Deflection	$\frac{dL}{L} 10^7$
0.055 amp.	1.0	29	0.75 mm	0.14
0.148	3.2	74	6.5	1.26
0.289	9.5	123	17.8	3.46
0.472	24.9	149	26.0	5.05
0.872	66.0	169	36.5	7.10
1.40	113.0	180	41.5	8.07
2.14	184.0	187	47.0	9.13
5.22	494.0	198	52.0	10.10

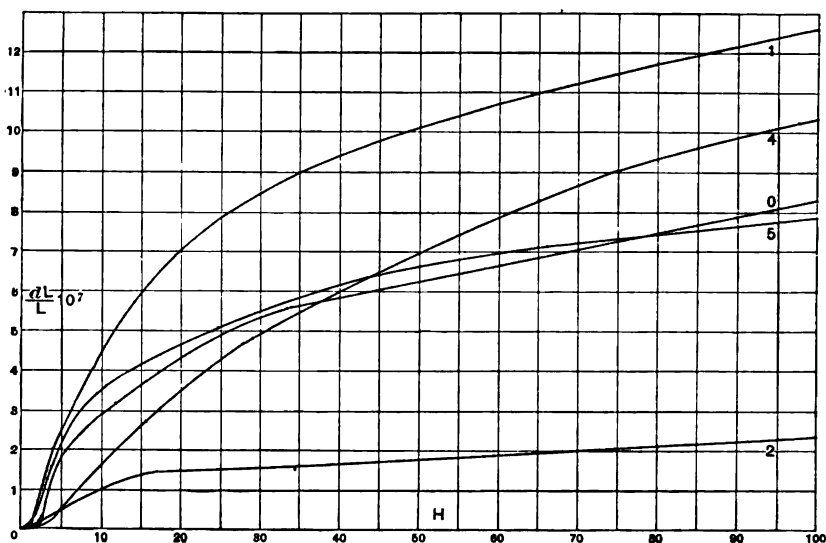
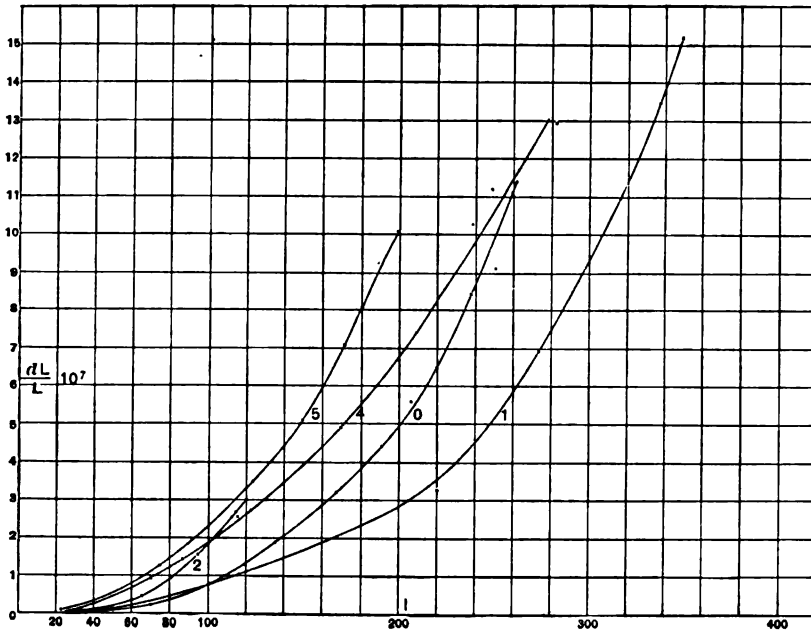


Fig. 7.—Curves for Magnetostriction.

case disappeared. It seems to be a fair conclusion to assume that the harder the alloy, the higher Young's modulus; but, as already mentioned, it was impossible to determine the latter, due to the mechanical imperfections of the rods.

Fig. 8.—Magnetostriction as a Function of I .**THERMOELECTRIC POWER.**

Bidwell's observation that the change of the thermoelectromotive force of iron and nickel is proportional to the magnetic expansion, corrected for the mechanical stress, led us to try some similar experiments with these Heusler alloys. Around their ends thin copper wires were wound tightly. They were then placed in the center of a glass tube just fitting inside the magnetizing coil (Fig. 9). Rub-

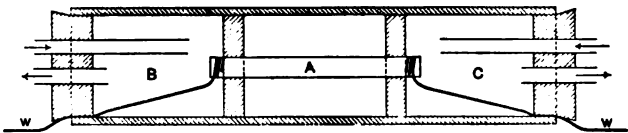


Fig. 9.—Apparatus for Thermoelectric Measurements.

ber stoppers were placed on the rods near the thermojunctions and the glass tube was closed at both ends by stoppers, through which two smaller tubes passed for the circulation of warm and cold water, respectively, in the chambers B and C. The thermoelectromotive

force produced by the difference of temperature was balanced by means of a potentiometer. A change in the emf. of one microvolt gave a galvanometer deflection of 2 mm. At first only moderate differences of temperature (50°) were used, but when not the slightest effect was observed on magnetizing the rods, we finally sent steam through the heating chamber, while ice-cold water flowed through the other. In this case also, though the thermoelectromotive force amounts to several millivolts we were unable to detect any change as large as 0.5 microvolt for any of the alloys, even in fields as large as 1,000 gauss. Bidwell found for iron and a temperature difference of about 85° a change as large as 15 microvolts in a field of 200 gauss in the case of nickel, and 25 microvolts. In connection with this, we may add that C. E. Mendenhall has informed us by letter that he was unable to find Kerr's phenomenon in a piece of Heusler alloy. We may therefore draw the interesting conclusion that in these alloys certain properties seem to be absent which have always been thought to be closely connected with magnetic substances.

A POCKET SPECTROPHOTOMETER.

By P. G. Nutting.

The instrument here described is very small and inexpensive and requires practically no adjustment or calibration. Its sensibility is ample for all but precision work, while its simplicity and compactness recommend it for general laboratory use. Aside from its extremely small size, the use of this combination of amici and nicol prisms is believed to be a novel feature in spectrophotometers.

As may be seen from Fig. 1, the instrument is essentially a pocket spectroscope with two nicols added, one before and one behind the

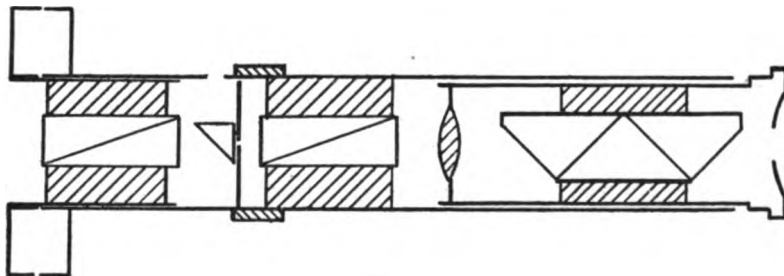


Fig. 1.

slit. The first nicol may be rotated about its axis, thus varying the intensity of the light coming from directly in front of the instrument but leaving the light coming from the side unaffected. The amount of this rotation is read off on the attached circular scale 5 cm in diameter.

For more accurate work or for comparing lined spectra (say an arc with a Nernst lamp) an ocular and ocular slit are added as in Fig. 2. This slit is kept of constant width (say 0.2 mm) and displaced laterally for settings on various spectrum lines.

The figures are of the actual dimensions of the instrument. Such an instrument was recently constructed for the Bureau of Standards by the firm of R. Fuess, of Steglitz, from designs by the

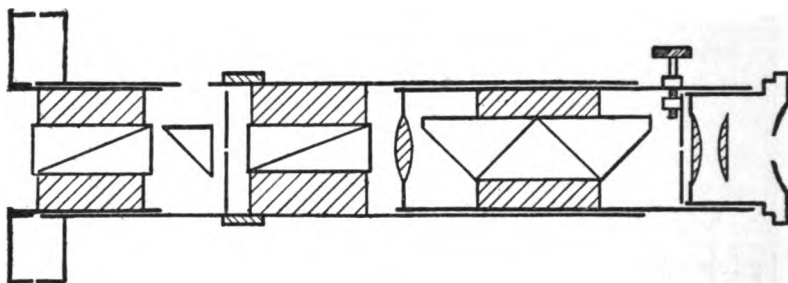


Fig. 2.

writer similar to Fig. 1, and to the excellence of their optical and mechanical workmanship is due the surprisingly high order of accuracy and sensibility of which this spectrophotometer is capable.

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PRELIMINARY MEASUREMENTS ON TEMPERATURE AND SELECTIVE RADIATION OF INCANDESCENT LAMPS.

By C. W. Waidner and G. K. Burgess.

The high efficiency attained in some recent metal filament lamps raises the question whether this is to be attributed to selective radiation or to a higher working temperature of the filament, or to both. As in this region of high temperatures almost no data are available at present, the results of some preliminary measurements on the temperature and selective radiation of a number of metal filament lamps are given in this paper.

METHOD OF MEASUREMENT.

The method of measurement is as follows:

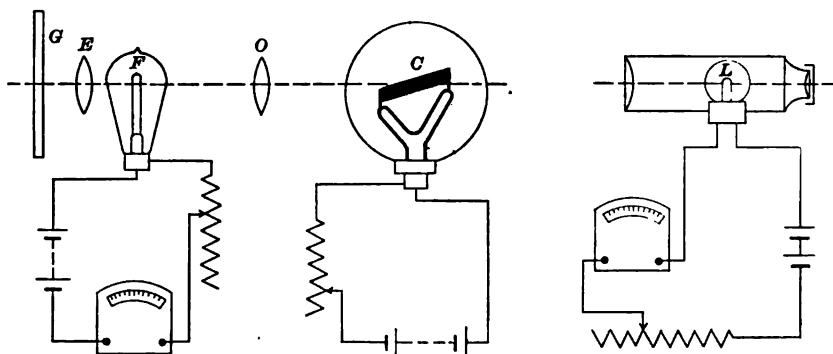


Fig. 1.

The filament *F* of the lamp under observation is mounted in front of the carbon ribbon *C*, which is itself within an evacuated globe, in such a way that, by means of the lenses *O* and *E*, the filament is seen superimposed on the carbon ribbon *C*. This ribbon may be heated electrically to any desired temperature, which is measured

by means of the Holborn-Kurlbaum optical pyrometer,¹ the principal of which is briefly as follows:

The current in the pyrometer lamp L is adjusted until the filament is of the same brightness as the incandescent body under observation. From a previous calibration by comparison with thermocouples and a black body source, the relation between the current in the filament and its equivalent black body temperature is known.

An observation at one temperature consists in setting the two filaments F and L to the same brightness as C , and measuring the currents in F and L . The temperatures of C and of F are then known from the calibration of L . From a series of such measurements at different temperatures the temperature-current curve for the filament F may be drawn.

Temperature Scale.—The lamp L being calibrated by comparison with a black body, when sighted on another incandescent body, reads not its true temperature, but something lower, by an amount depending on the emissive power of the body. By the term "*black body temperature*," is therefore meant the temperature at which a black body would send out radiation of the same intensity as that from the object observed for a given wave length. A body at a given temperature will in general have a different black body temperature for each color, that for red being lower than for green or blue. In this paper the term *temperature*, unless otherwise qualified, is taken to mean the black body temperature centigrade as given by an optical pyrometer using red light, $\lambda = 0.66\mu$.

To study the selective radiation for different colors, the filament F was calibrated with red, green, and blue light, obtained by interposing monochromatic glasses at G .

The measurements were made in the range 700° – 1850° C., which was the safe upper limit of the carbon ribbon C . Higher temperatures of F were then obtained by extrapolation of the current-temperature relation. As to the reliability of such extrapolation, we have found that lamps calibrated to 1300° were still in agreement to within 1° or 2° when extrapolated 300° , and further that these extrapolations are in satisfactory agreement with other methods of measuring temperature.

¹The subject of Optical Pyrometry is treated at length in Bulletin of Bureau of Standards, 1, p. 189–254; 1905.

Precision of Method.—As an illustration of the results attainable by this method we give in Table 1 the measurements made on a 100-volt metal filament lamp, very probably tungsten.

TABLE 1.

Calibration of Metal Filament Lamp, No. 11 (red radiation).

Current in Filament.	Observed Temperature of Filament.	Calculated Temperature.	Observations Calculated.
0.1302	847°	848°2	—1°2
0.1510	921°	919°2	+1°8
0.1895	1049°	1051°2	—2°2
0.2270	1166°	1167°2	—1°2
0.2625	1267°	1269°9	—2°9
0.3144	1413°	1410°7	+2°3
0.4013	1621°	1625°2	—4°2
0.4561	1750°	1750°2	—0°2
0.4812	1811°	1805°7	+5°3

These observations satisfy the following equation:

$I = -0.0285 + 0.0001041t + 0.0000000990t^2$ for red radiation ($\lambda = 0.66\mu$), where I is the current in amperes and t is the temperature.

SELECTIVE RADIATION.

Regarding the significance of selective radiation and its bearing upon the problem in hand, the radiation of platinum for red, green, and blue, as compared with black body radiation, may serve as an extreme illustration. In the following table are grouped some results obtained from measurements of the black body temperatures of platinum for red, green, and blue at various known temperatures.

TABLE II.

Radiation from Platinum.

Actual Temperature of Platinum	Black Body Temperatures		
	Red $\lambda = 0.66\mu$	Green $\lambda = 0.55\mu$	Blue $\lambda = 0.47\mu$
1100	1008	1029	
1400	1255	1285	1300
1700	1505	1545	1575

We may interpret these results as follows:

As a first approximation suppose that the platinum is at a true temperature of 1700° , then its radiation for green light is equal in intensity to the radiation of a black body at 1545° . The platinum radiation will be somewhat greater than that of this black body for blue light (viz equivalent to a black body at 1575°) and somewhat less for red (viz 1505°). The total luminous radiation of the two is therefore not very different, and as the maximum sensibility of the eye is in the green their photometric appearance will be very nearly alike. Now the maximum of energy for both is in the infra-red, and for these long wave lengths the black body temperature of the platinum falls still more behind that of the black body. It will be seen therefore that the energy of luminous radiation is distributed more favorably for platinum than for a black body.

In the calibration of the carbon filament pyrometer lamps against a black body, no appreciable difference (less than 2° C.) could be detected in the current-temperature calibration equation using red, green, and blue light. It does not follow from this, however, that the radiation from carbon is the same as that from a black body; on the contrary carbon is known to depart considerably from ideal blackness, although in the visible spectrum it shows no appreciable evidence of selective radiation, and for this reason is sometimes called a "gray body."

TUNGSTEN LAMPS.

Four tungsten filaments gave the following current-temperature

TABLE III.

Equations for Tungsten Lamps.

Lamp No.	Color	Equation
1	Red	$I = 1.008 - 0.000\ 391\ t + 0.000\ 000\ 548\ t^2$
5	Red	$I = 1.344 - 0.000\ 396\ t + 0.000\ 000\ 518\ t^2$
10	Red	$I = 0.802 - 0.000\ 437\ t + 0.000\ 000\ 490\ t^2$
10	Green	$I = 0.775 - 0.000\ 390\ t + 0.000\ 000\ 465\ t^2$
10	Blue	$I = 0.644 - 0.000\ 225\ t + 0.000\ 000\ 411\ t^2$
11	Red	$I = -.028 + 0.000\ 104\ t + 0.000\ 000\ 099\ 0\ t^2$
11	Green	$I = -.060 + 0.000\ 152\ t + 0.000\ 000\ 078\ 1\ t^2$
11	Blue	$I = -.056 + 0.000\ 148\ t + 0.000\ 000\ 077\ 8\ t^2$

relation, where I is the current in amperes in the filament and t is its corresponding black body temperature for radiation (color) studied.

All but lamp No. 11 were low voltage experimental lamps, designed with a view to studying the behavior of tungsten at high temperatures. The experimental data for lamp No. 11 are given below. (See Table I et seq.) At normal voltage (100 v.) the current in this lamp was 0.644 ampere, from which the corresponding temperature given by its equation is 2135° C., indicating a true temperature of the filament of about 2300° C. (See Table IV.) At 100 volts the candlepower of this lamp was 68 and its consumption 0.95 watt per candle (mean horizontal), from data furnished by Dr. Hyde, of this Bureau. Lamp No. 10 was burned at a temperature of 2400° C. (about 2570° C. true temperature) for $1\frac{1}{2}$ hours when it burned out.

Selective Radiation.—The selective radiation of tungsten was studied as in the case of platinum by measuring black body temperatures of the filament for red, green, and blue light. The results as given by the equations of Table III are shown in Table IV.

TABLE IV.

Selective Radiation of Tungsten.

Lamp No.	Black Body Temperatures			Approximate True Temperatures
	Red ($\lambda=0.66$)	Green ($\lambda=0.55$)	Blue ($\lambda=0.47$)	
10	1300° C	1310° C	1319° C	1355° C
11	"	1311°	1319°	1355°
10	1700°	1714°	1723°	1770°
11	"	1724°	1734°	1800°
10	2100°	2123°	2141°	2220°
11	"	2146°	2161°	2280°
10	2500°	2532°	2565°	2690°
11	"	2576°	2594°	2780°
10	2900°	2943°	2994°	3190°

The appearance of the filament of lamp No. 11 when cold was more polished than that of No. 10, and the table shows No. 11 to

act the more like a bright metal such as platinum. Another lamp whose filament resembled that of No. 10 gave sensibly identical values with the latter. The last column in Table IV, giving the approximate values of the actual temperatures of the tungsten filaments, is obtained by adding to the black body temperature for blue light twice the difference between the red and blue readings, a relation found to hold fairly well for platinum, which shows this selective effect much more than does tungsten. (See Table II.)

A filament, the composition of which was stated to be 30% tungsten and 70% zirconium nitrate, gave practically the same selective radiation as tungsten. When run at a temperature somewhat above the normal working temperature of the tungsten lamp, a rapid deterioration of the filament took place, accompanied by the formation of an iridescent deposit on the bulb.

Melting Point of Tungsten.—By noting the current required to burn out a tungsten lamp and substituting in the proper current-temperature equation (Table III) an idea of the melting point of tungsten may be obtained. Lamps 1 and 5 were burned out, at temperatures of 2950° and 2850° C. respectively, which indicates a mean true temperature of about 3200° C. for the melting point of tungsten. Both filaments formed shiny beads, indicating a true melt and not a disintegration due to evaporation as in the case of carbon filaments. There was no appreciable deposit in the bulbs after burning out. It would appear that tungsten has the highest melting point yet measured.

TANTALUM LAMPS.

We have thus far examined only two tantalum lamps, whose current-temperature equations are given in Table V.

TABLE V.

Equations for Tantalum Lamps.

Lamp No.	Color	Equation
8	Red	$I = -0.0125 + 0.000\ 053\ 8\ t + 0.000\ 000\ 084\ 0\ t^2$
8	Green	$I = -0.0392 + 0.000\ 090\ 1\ t + 0.000\ 000\ 068\ 7\ t^2$
8	Blue	$I = -0.0607 + 0.000\ 125\ t + 0.000\ 000\ 053\ 1\ t^2$
9	Red	$I = -0.0176 + 0.000\ 061\ 7\ t + 0.000\ 000\ 081\ 0\ t^2$

The filament of No. 9 had been broken, and rewelded by shaking. No. 8 was the ordinary 110 volt lamp. At 110 volts the current was 0.380 ampere, indicating a temperature of 1865°C ., or a true temperature of about 2000°C . (See Table VI.) This lamp was heated for one hour at the normal temperature of the tungsten lamp, 2135°C ., after which a recalibration showed a marked increase in temperature (about 2%) for a given current. A further two hours heating at the same temperature showed a further rise of 1%. After standing ten days it recovered almost completely, the temperature of normal burning (110 v.) as determined from a new calibration being 1870°C . Lamp No. 9 was burned at 2200°C . for seven hours and showed the same phenomena, but to a much greater extent, and there was a very marked blackening of its bulb.

The increase in efficiency of these metal filament lamps in the early stages of burning is well known. The above experiments indicate a rise in temperature during this stage of the burning. Sufficient data are not yet at hand to determine whether this is due to a smoothing of the surface as has been suggested or to an improvement in the vacuum with burning. If the increase in efficiency is due to an increase in the polish of the surface, and therefore in the selectivity of the radiation, it would probably result in further separating the red and blue calibration curves.

Selective Radiation.—In Table VI are given the results of measurements on the selective radiation of tantalum, as obtained with lamp No. 8.

TABLE VI.
Selective Radiation of Tantalum.

Black Body Temperatures			Approximate True Temperature
Red $\lambda=0.66$	Green $\lambda=0.55$	Blue $\lambda=0.47$	
1300°C	1320°C	1330°C	1360°C
1700°	1727°	1752°	1800°
2100°	2147°	2198°	2300°

CARBON LAMPS.

For the sake of comparison with the metal filament lamps a study was made of the temperature behavior of some of the ordinary types

of carbon filament lamps, including 4, 3.5, and 3.1 watt lamps. The current-temperature equations of five of these lamps are given in Table VII.

TABLE VII.

Equations of Carbon Filament Lamps.

Lamp No.	Type		Equation
	Volts	Watts	
3	50	4	$I = 0.156 - 0.000\ 223\ t + 0.000\ 000\ 532\ t^2$
4	50	4	$I = .092 - 0.000\ 086\ 0\ t + 0.000\ 000\ 460\ t^2$
6	50	4	$I = .087 + 0.000\ 039\ 2\ t + 0.000\ 000\ 388\ t^2$
7	50	4	$I = .166 - 0.000\ 00\ 140\ t + 0.000\ 000\ 394\ t^2$
7	After 2200°		$I = .066 + 0.000\ 109\ t + 0.000\ 000\ 352\ t^2$
13	110	3.1	$I = .067 - 0.000\ 091\ 6\ t + 0.000\ 000\ 156\ t^2$

With a view to determining the maximum temperature attainable with carbon filaments, several of these lamps were burned out by quickly increasing the current.

Owing to the rapid deterioration of the carbon filament the maximum temperature that can be attained depends on the rapidity with which the temperature is raised, the thickness and condition of the filament, etc. For these reasons it is impossible to state with precision the temperature at which the filament finally breaks down as the calibration equation no longer applies, but serves, nevertheless, to define a lower limit of the temperature of destructive disintegration, which varied between 2500° and 2800° C. for these lamps. Already at the normal working temperature of the tungsten (2135°) the carbon lamp shows rapid deterioration.

The normal burning temperatures of lamps 3, 4, 6, and 7 ranged from 1695° to 1720°. Lamp No. 7 was burned at 2200° for 15 minutes and on recalibration showed a normal temperature of 1670°, or 40° lower than before. It was again heated one hour at 2200°, when it broke down at the pasted junction. The bulb showed considerable blackening and the resistance of the lamp rose from 36.7 to 41.5 ohms.

NORMAL TEMPERATURES.

The following Table, VIII, gives the normal burning temperatures of both the metal filament and carbon lamps examined.

TABLE VIII.

Normal Burning Temperatures.

Type of lamp	Watts	Volts	Observed Black Body Temperatures (Red)	Approximate True Temperature
Carbon	4.0	50	1710° C	1800° C
"	3.5	118	1760°	1850°
"	3.1	118	1860°	1950°
Tantalum	2.0	110	1865°	2000°
Tungsten	1.0	100	2135°	2300°

GENERAL DISCUSSION.

At a given true temperature the total energy of thermal radiation, as well as the energy of radiation for every wave length, emitted per unit area by a black body, is greater than that of any other known body. No conclusive experimental evidence has yet been brought forward in contradiction to this general statement. On account of the very large proportion of the energy of total radiation that exists as the longer wave lengths of the infra-red portion of the spectrum that do not excite the sensation of light in the eye, a black body is an inefficient luminous radiator.

Of all metals that can be raised to even a moderately high temperature (say 1500° or more) platinum departs farthest from black body radiation. For a given true temperature it radiates less total energy, and a larger proportion of this energy exists in the form of the shorter wave lengths which excite the sensation of light. This is the sense in which the term *selective radiation* is here used. If platinum would stand the high temperatures at which these modern metal filament lamps can be worked, it would therefore have an appreciably higher efficiency than they have. In this sense all solid substances show, in varying degrees, selective radiation, and all are more efficient luminous radiators than a black body. For these reasons, carbon, which is one of the closest approximations to a black

body, is a less efficient luminous radiator than the metals. It should be remembered, however, that at the same true temperature a carbon filament would emit more light than these metal filaments, although, on account of the greater selective radiation of the metals as shown above, it would be less efficient.

In much of the literature on this subject the marked gain in efficiency of the metal filament lamps is attributed almost entirely to selective radiation, and it is often implied that the radiation is not only selective in the sense above discussed, but similar to that of a gas, which, when electrically excited, can emit strongly in one region of the spectrum while the energy of radiation may be entirely absent for large regions. In support of this view the statement is often made that the character of the light from the tungsten filament is more greenish in appearance than from a carbon filament or from a tantalum filament.

In this connection it should be stated that a few preliminary measurements made on the Nernst glower at low temperatures seem to show that there is an appreciable increase in the selective radiation for green light, the black body temperature for green light being almost or quite as high as for blue, which is in agreement with the experiments of Kurlbaum and Schulze.² The nature of the conduction and of the chemical processes here involved is as yet but little understood. The undue increase in the emission for green light might suggest the combined effects of radiation from a solid and from a gas. The experiments of Kurlbaum and Schulze show that this effect almost disappears at higher temperatures, and the important fact remains that it is not this type of selective radiation that contributes materially to the efficiency of any of the metal filament lamps, unless there be a considerable effect of this kind in the infra-red.

The measurements above cited show that tantalum is more selective in its radiation than is tungsten, and in all probability would be more efficient than tungsten at the same true temperature. The great gain in efficiency in the tungsten lamp over the tantalum lamp must therefore be attributed to the very much higher temperature at which the tungsten can be worked continuously. Likewise the

² Kurlbaum and Schulze: *Verhandl. d. Deutsch. Phys. Gesell.* 5, p. 428; 1903.

marked increase in efficiency shown by both tantalum and tungsten lamps over carbon filament lamps is to some extent due to selective radiation, but is rendered possible to a greater extent by the fact that they can be operated at a higher working temperature. The marked gain in efficiency resulting from an increase in the temperature is at once evident from the fact, that at the working temperature of these lamps the intensity of the light emitted varies about as the twelfth power of the temperature, while the energy supplied to the filament varies as a much lower power of the temperature, something of the order of the fifth, varying with the material of which the filament is constructed and the nature of its surface. This is also illustrated by some measurements made by Dr. Lederer³ on Osmin lamps, from which it follows that the light increases as the 4.4 power of the voltage, while the energy consumption varies only as the 1.5 power of the voltage.

If a very considerable part of the gain in efficiency in the metal filament over carbon filament lamps is to be attributed to the higher working temperature at which they may be operated it may at first sight seem that this conclusion is at variance with the relatively high efficiency of these metal filament lamps in comparison with electric arc lights where the luminous radiation comes from a source whose temperature is over 1500° higher. But in the arc light the loss of energy by the conduction and radiation of the carbons and by convective circulation is so great that this fact is readily explained.

We are indebted to Mr. J. W. Howell for the carbon ribbon lamps, and to Mr. J. A. Heany for most of the tungsten lamps.

³ Libesny: *Electrotechnik und Maschinenbau*, 21, p. 438; 1906.

REVISION OF THE FORMULÆ OF WEINSTEIN AND STEFAN FOR THE MUTUAL INDUCTANCE OF COAXIAL COILS.

By Edward B. Rosa.

The only formulæ for the mutual inductance of two coaxial coils of rectangular cross section that have been proposed which take account of differential coefficients higher than the second are those of Weinstein¹ and Stefan.² The formulæ of Rowland,³ Rayleigh,⁴ and Lyle⁵ take account of second differentials only, and while sufficiently accurate for coils of relatively small section and considerable distance apart are not so accurate for coils of larger cross section, or for coils of small section if they are relatively near each other. The formulæ of Weinstein and Stefan seem to have been very little employed, and so far as I know have not been critically examined or compared with one another. The former of these two formulæ is given on page 340, of this paper, equation (14)—the latter is (21) of page 342.

Weinstein derived his formula by starting with Maxwell's expression in elliptic integrals for the mutual inductance of two coaxial circles (equation 13 below) and differentiating it, as Rowland did, carrying the operation, however, to the fourth order of differentials. Stefan, on the other hand, took Maxwell's second expression (equation 25 below), where the mutual inductance of two coaxial circles is expressed in a converging series, and differentiated it, or certain terms of it, to the sixth order. Omitting the terms in the latter depending on differentials higher than the fourth order, these two

¹ Wied. Annalen, **21**, p. 329; 1884.

² Wied. Annalen, **22**, p. 115; 1884.

³ American Journal of Science, **15**; 1878.

⁴ Maxwell, Vol. II, Chapter XIV, Appendix II.

⁵ Phil. Mag., **8**, p. 310; 1902.

formulae should give the same results. They do not, however, agree as closely as they ought, and examination shows that Weinstein's is very nearly exact when the coils are at considerable distance from one another, but less accurate when they are near. Stefan's, on the other hand, is very accurate when the coils are near but less accurate when they are far apart, the relative error increasing with the distance. The error in either case is small, and yet too large for precision work, and larger than can be explained by the terms of higher degree omitted in deriving the formulae. In going over Weinstein's work, and in deriving Stefan's formula anew (Stefan gives only his result, without any indication of the process by which it was derived)

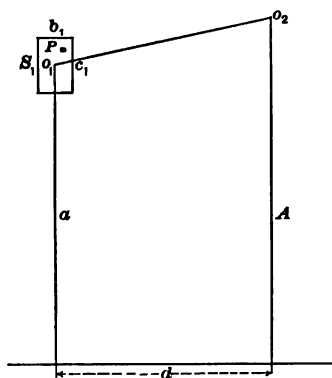


Fig. 1

I have found errors in both formulae. These being corrected they agree as well as could be expected over a wide range of distances and for coils of considerable cross section. Stefan's formula can be put into a form that is easy for calculation, being much more convenient than the other less accurate formulae. Stefan's and Weinstein's formulae are, however, limited by the condition that the two coils are of equal radii, whereas Rayleigh's and Lyle's apply also to coils of unequal radii.

As formulae for the accurate calculation of the mutual inductance of parallel coils are of fundamental importance, and as these formulae of Weinstein and Stefan as revised and corrected are the most accurate formulae in existence (for coils of equal radii), I shall give somewhat fully the derivation of Stefan's and the revision of Weinstein's, and shall put them into different forms, which are more convenient for calculation than the forms in which they were originally given.

MUTUAL INDUCTANCE OF A CIRCLE AND A COIL.

Let O_2 be a circle of radius A and S_1 be a coil of mean radius a , axial breadth b_1 , and radial depth c_1 , the distance between the mean plane of the coil and the plane of the circle being d . We wish to find first the mutual inductance of the circle O_2 and the coil S_1 .

One way to proceed would be to take the expression for the mutual inductance of the circle O_1 and a second parallel circle P of cross sectional area $dx dy$ within the section of the coil and integrate this over the area of the section S_1 . This is a difficult and complicated operation, and has never been done. Maxwell,⁶ however, shows how to get an approximate result more easily by means of Taylor's theorem, and this result can be made very accurate by taking a sufficient number of terms.

If $f(x, y) = u$, Taylor's theorem gives

$$\begin{aligned} f(x+h, y+k) = & u + \left(h \frac{du}{dx} + k \frac{du}{dy} \right) + \frac{1}{1.2} \left(h^2 \frac{d^2u}{dx^2} + 2hk \frac{d^2u}{dx dy} + k^2 \frac{d^2u}{dy^2} \right) \\ & + \frac{1}{1.2.3} \left(h^3 \frac{d^3u}{dx^3} + 3h^2k \frac{d^3u}{dx^2 dy} + 3hk^2 \frac{d^3u}{dx dy^2} + k^3 \frac{d^3u}{dy^3} \right) \\ & + \dots \dots \dots + \frac{1}{n!} \left(h \frac{d}{dx} + k \frac{d}{dy} \right)^n u \quad (1) \end{aligned}$$

Applying this to the case of the circle and the coil, h and k would be the coordinates of the point P with O_1 as origin, u would be the mutual inductance of the circle O_1 and the circle O_2 , and $f(x+h, y+k)$ would be the mutual inductance of the circle O_1 and the circle P . To get the mutual inductance of the circle O_1 and the coil S_1 it will be necessary to integrate this expression over the area of S_1 and divide by the area b_1c_1 .

Using Maxwell's notation, the mean value $\bar{P}$ of any function P of x and y is

$$\bar{P} = \frac{1}{xy} \int_{-\frac{x}{2}}^{+\frac{x}{2}} \int_{-\frac{y}{2}}^{+\frac{y}{2}} P dx dy \quad (2)$$

This mean value is, of course, the mean for the region within the limits of integration, and if this operation be applied to the expression for mutual inductance of the two circles O_1 and P we can obtain an expression for the mutual inductance of O_1 and the coil S_1 of section b_1c_1 . Writing M_0 for the mutual inductance of O_1 and O_2 ,

⁶Electricity and Magnetism, Vol. II, § 700.

M' for the mutual inductance of P and O_1 , and x and y in place of h and k , we have in place of (1):

$$\begin{aligned}
 M' = & M_0 + \left(x \frac{dM_0}{dx} + y \frac{dM_0}{dy} \right) + \frac{1}{1.2} \left(x^2 \frac{d^2 M_0}{dx^2} + 2xy \frac{d^2 M_0}{dxdy} + y^2 \frac{d^2 M_0}{dy^2} \right) \\
 & + \frac{1}{3!} \left(x^3 \frac{d^3 M_0}{dx^3} + 3x^2 y \frac{d^3 M_0}{dx^2 dy} + 3xy^2 \frac{d^3 M_0}{dx dy^2} + y^3 \frac{d^3 M_0}{dy^3} \right) \\
 & + \frac{1}{4!} \left(x^4 \frac{d^4 M_0}{dx^4} + 4x^3 y \frac{d^4 M_0}{dx^3 dy} + 6x^2 y^2 \frac{d^4 M_0}{dx^2 dy^2} + 4xy^3 \frac{d^4 M_0}{dx dy^3} + y^4 \frac{d^4 M_0}{dy^4} \right) \\
 & + \frac{1}{5!} \left(\dots \dots \dots \right) \\
 & + \frac{1}{6!} \left(x^6 \frac{d^6 M_0}{dx^6} + \dots + 15x^4 y^2 \frac{d^6 M_0}{dx^4 dy^2} + \dots + 15x^2 y^4 \frac{d^6 M_0}{dx^2 dy^4} \right. \\
 & \left. + \dots + y^6 \frac{d^6 M_0}{dy^6} \right) + \dots
 \end{aligned} \tag{3}$$

The subscript attached to M in the derivatives above indicates that the values of x and y corresponding to the circle O_1 at the center of the section are to be inserted in the derivatives *after* differentiation.

To obtain M_1 the mean value of M' over the area of the section, which will be the mutual inductance of O_1 and the coil S_1 , we must apply the operation indicated above in (2) to each term of (3). It will be noticed that every term containing an odd power of x or y will become zero when the limits are inserted, the derivatives $\frac{d^2 M_0}{dx^2}$ etc., being constant. Applying this operation to the five terms in (3) which do not reduce to zero gives the following results:

$$\begin{aligned}
 \frac{1}{2} \int_{-\frac{x}{2}}^{+\frac{x}{2}} \int_{-\frac{y}{2}}^{+\frac{y}{2}} x^2 dx dy &= \frac{x^3 y}{3 \cdot 2^2} = \frac{x^3}{24} xy \\
 \frac{1}{4!} \int \int x^4 dx dy &= \frac{2x^5 y}{4! 5 \cdot 2^2} = \frac{x^5}{1920} xy \\
 \frac{6}{4!} \int \int x^2 y^2 dx dy &= \frac{24x^3 y^3}{4! 3^2 \cdot 2^2} = \frac{x^3 y^3}{576} xy
 \end{aligned}$$

$$\frac{1}{6!} \iint x^6 dx dy = \frac{2x^7 y}{6! \cdot 2} = \frac{x^7 y}{120 \cdot 2688}$$

$$\frac{15}{6!} \iint x^4 y^3 dx dy = \frac{4x^5 y^3}{6! \cdot 2^3} = \frac{x^5 y^3}{120 \cdot 384}$$

Four other terms having even powers of x and y give similar results, while nine terms in odd powers and those not written reduce to zero.

Substituting these values in equation (3) and writing $\frac{d^3 M_0}{da^3}$ for $\frac{d^3 M_0}{dy^3}$ etc., and b_1 and c_1 as the limits of integration in place of x and y , respectively, we have:

$$\begin{aligned} M_1 = M_0 &+ \frac{b_1^2}{24} \frac{d^3 M_0}{dx^3} + \frac{c_1^2}{24} \frac{d^3 M_0}{da^3} + \frac{b_1^4}{1920} \frac{d^5 M_0}{dx^5} + \frac{c_1^4}{1920} \frac{d^5 M_0}{da^5} \\ &+ \frac{b_1^2 c_1^2}{576} \frac{d^5 M_0}{dx^3 da^3} + \frac{b_1^6}{120 \cdot 2688} \frac{d^7 M_0}{dx^7} + \frac{c_1^6}{120 \cdot 2688} \frac{d^7 M_0}{da^7} \\ &+ \frac{b_1^4 c_1^4}{120 \cdot 384} \frac{d^7 M_0}{dx^5 da^5} + \frac{b_1^2 c_1^2}{120 \cdot 384} \frac{d^7 M_0}{dx^3 da^3} + \dots \dots \dots (4) \end{aligned}$$

MUTUAL INDUCTANCE OF TWO COILS.

Equation (4) gives the mutual inductance of the circle O_1 and the coil of section $b_1 c_1$ and mean radius a . But we wish to obtain the mutual inductance of two coils of rectangular section, and hence the above operation must be applied again.

M_0 being the mutual inductance of circles O_1 and O_2 at the centers of the sections S_1 and S_2 , and M_1 being the mutual inductance of coil S_1 and circle O_2 , let M_2 be the mutual inductance of coil S_1 and coil S_2 . We can now obtain M_2 from M_1 in the same way that we have obtained M_1 from M_0 . In this case the variable coordinates A and x of the circle P_2 will appear in the differential coefficients, A occurring where

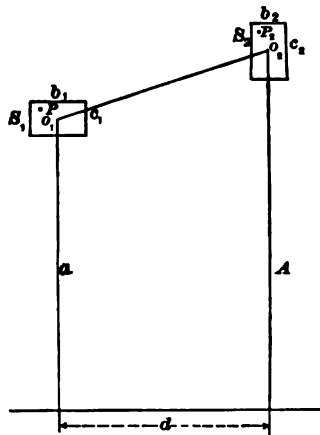


Fig. 2

a appears in (4), and M_1 replacing M_0 , while the limits of integration are b_2 and c_2 . Thus we obtain

$$\begin{aligned} M_2 = M_1 &+ \frac{b_1^2}{24} \frac{d^2 M_1}{dx^2} + \frac{c_1^2}{24} \frac{d^2 M_1}{dA^2} + \frac{b_1^4}{1920} \frac{d^4 M_1}{dx^4} + \frac{c_1^4}{1920} \frac{d^4 M_1}{dA^4} \\ &+ \frac{b_1^2 c_1^2}{576} \frac{d^2 M_1}{dx^2 dA^2} + \frac{b_1^6}{120 \cdot 2688} \frac{d^6 M_1}{dx^6} + \frac{c_1^6}{120 \cdot 2688} \frac{d^6 M_1}{dA^6} \\ &+ \frac{b_1^2 c_1^4}{120 \cdot 384} \frac{d^2 M_1}{dx^2 dA^4} + \frac{b_1^4 c_1^2}{120 \cdot 384} \frac{d^2 M_1}{dx^4 dA^2} + \dots \quad (5) \end{aligned}$$

We must now substitute the value of M_1 given by equation (4) in each term of (5), omitting derivatives higher than the sixth.

Thus:

$$\begin{aligned} M_2 = M_0 &+ \frac{b_1^2}{24} \frac{d^2 M_0}{dx^2} + \frac{c_1^2}{24} \frac{d^2 M_0}{dA^2} + \frac{b_1^4}{1920} \frac{d^4 M_0}{dx^4} + \frac{c_1^4}{1920} \frac{d^4 M_0}{dA^4} \\ &+ \frac{b_1^2 c_1^2}{576} \frac{d^2 M_0}{dx^2 dA^2} + \frac{b_1^6}{120 \cdot 2688} \frac{d^6 M_0}{dx^6} + \frac{c_1^6}{120 \cdot 2688} \frac{d^6 M_0}{dA^6} \\ &\quad + \frac{b_1^2 c_1^4}{120 \cdot 384} \frac{d^2 M_0}{dx^2 dA^4} \\ &+ \frac{b_1^4 c_1^2}{120 \cdot 384} \frac{d^2 M_0}{dx^4 dA^2} + \frac{b_1^2}{24} \frac{d^2 M_0}{dx^2} + \frac{b_1^2 b_1^2}{576} \frac{d^2 M_0}{dx^4} + \frac{b_1^2 c_1^2}{576} \frac{d^2 M_0}{dx^2 dA^2} \\ &+ \frac{c_1^4 b_1^2}{24 \cdot 1920} \frac{d^4 M_0}{dx^2 dA^4} + \frac{b_1^4 b_1^2}{24 \cdot 1920} \frac{d^4 M_0}{dx^6} + \frac{b_1^2 b_1^2 c_1^2}{24 \cdot 576} \frac{d^4 M_0}{dx^4 dA^2} + \frac{c_1^2}{24} \frac{d^2 M_0}{dA^2} \\ &+ \frac{c_1^2 c_1^2}{576} \frac{d^2 M_0}{dA^2 dA^2} + \frac{b_1^2 c_1^2}{576} \frac{d^2 M_0}{dx^2 dA^2} + \frac{c_1^4 c_1^2}{24 \cdot 1920} \frac{d^4 M_0}{dA^4 dA^2} \\ &\quad + \frac{b_1^2 c_1^2 c_1^2}{24 \cdot 576} \frac{d^2 M_0}{dx^2 dA^2 dA^2} \\ &+ \frac{b_1^4 c_1^2}{24 \cdot 1920} \frac{d^4 M_0}{dx^4 dA^2} + \frac{c_1^4}{1920} \frac{d^4 M_0}{dA^4} + \frac{c_1^2 c_1^4}{24 \cdot 1920} \frac{d^4 M_0}{dA^2 dA^4} \\ &\quad + \frac{b_1^2 c_1^4}{24 \cdot 1920} \frac{d^2 M_0}{dx^2 dA^4} \\ &+ \frac{b_1^4}{1920} \frac{d^4 M_0}{dx^4} + \frac{b_1^4 c_1^2}{24 \cdot 1920} \frac{d^4 M_0}{dx^4 dA^2} + \frac{b_1^2 b_1^4}{24 \cdot 1920} \frac{d^4 M_0}{dx^6} \\ &\quad + \frac{b_1^2 c_1^2}{576} \frac{d^2 M_0}{dx^2 dA^2} \\ &+ \frac{b_1^2 c_1^2 c_1^2}{24 \cdot 576} \frac{d^2 M_0}{dx^2 dA^2 dA^2} + \frac{b_1^2 b_1^2 c_1^2}{24 \cdot 576} \frac{d^2 M_0}{dx^4 dA^2} + \frac{c_1^6}{120 \cdot 2688} \frac{d^6 M_0}{dA^6} \\ &+ \frac{b_1^6}{120 \cdot 2688} \frac{d^6 M_0}{dx^6} + \frac{b_1^2 c_1^4}{120 \cdot 384} \frac{d^2 M_0}{dx^2 dA^4} + \frac{b_1^4 c_1^2}{120 \cdot 384} \frac{d^2 M_0}{dx^4 dA^2} + \dots \quad (6) \end{aligned}$$

The first 10 terms of equation (6) come from substituting for M_1 in (5) its value given by (4). The last four terms are the last four terms of (5). The remaining terms come from substituting in the first five differential terms of (5) the value of M_1 in (4). Collecting like terms together equation (6) may be written

$$\begin{aligned}
 M_2 = M_0 + \frac{1}{24} & \left\{ (b_1^2 + b_2^2) \frac{d^2 M_0}{dx^2} + c_1^2 \frac{d^2 M_0}{da^2} + c_2^2 \frac{d^2 M_0}{dA^2} \right\} \\
 & + \frac{1}{1920} \left\{ (b_1^4 + b_2^4) \frac{d^4 M_0}{dx^4} + c_1^4 \frac{d^4 M_0}{da^4} + c_2^4 \frac{d^4 M_0}{dA^4} \right\} \\
 & + \frac{1}{576} \left\{ b_1^2 b_2^2 \frac{d^4 M_0}{dx^4} + c_1^2 c_2^2 \frac{d^4 M_0}{da^2 dA^2} \right\} \\
 & + \frac{1}{576} (b_1^2 + b_2^2) \left\{ c_1^2 \frac{d^4 M_0}{dx^2 da^2} + c_2^2 \frac{d^4 M_0}{dx^2 dA^2} \right\} \\
 & + \frac{1}{120 \cdot 384} \left\{ \left(\frac{b_1^6}{7} + b_1^4 b_2^2 + b_1^2 b_2^4 + \frac{b_2^6}{7} \right) \frac{d^6 M_0}{dx^6} \right\} \\
 & + \frac{1}{120 \cdot 384} \left\{ \frac{c_1^6}{7} \frac{d^6 M_0}{da^6} + \frac{c_2^6}{7} \frac{d^6 M_0}{dA^6} + c_1^2 c_2^4 \frac{d^6 M_0}{da^2 dA^4} + c_1^4 c_2^2 \frac{d^6 M_0}{da^4 dA^2} \right\} \\
 & + \frac{1}{120 \cdot 384} \left\{ (b_1^2 + b_2^2) c_1^4 \frac{d^6 M_0}{dx^2 da^4} + (b_1^2 + b_2^2) c_2^4 \frac{d^6 M_0}{dx^2 dA^4} \right\} \\
 & + \frac{1}{120 \cdot 384} \left\{ (b_1^4 + b_2^4) c_1^2 \frac{d^6 M_0}{dx^4 da^2} + (b_1^4 + b_2^4) c_2^2 \frac{d^6 M_0}{dx^4 dA^2} \right\} \\
 & + \frac{5}{120 \cdot 576} \left\{ (b_1^2 + b_2^2) c_1^2 c_2^2 \frac{d^6 M_0}{dx^2 da^2 dA^2} \right. \\
 & \quad \left. + b_1^2 b_2^2 \left(c_1^2 \frac{d^6 M_0}{dx^4 da^2} + c_2^2 \frac{d^6 M_0}{dx^4 dA^2} \right) \right\} \quad (7)
 \end{aligned}$$

Equation (7) is very much simplified by taking the two coils of equal radii, and equal section. Thus, let

$$A = a, \quad b_1 = b_2 = b, \quad c_1 = c_2 = c$$

We then have

$$\begin{aligned}
 M = M_0 + \frac{1}{12} & \left(b^2 \frac{d^2 M_0}{dx^2} + c^2 \frac{d^2 M_0}{da^2} \right) + \frac{1}{360} \left(b^4 \frac{d^4 M_0}{dx^4} + c^4 \frac{d^4 M_0}{da^4} \right) \\
 & + \frac{1}{144} \left(b^2 c^2 \frac{d^4 M_0}{dx^2 da^2} \right) \\
 & + \frac{1}{120 \cdot 504} \left(3b^6 \frac{d^6 M_0}{dx^6} + 3c^6 \frac{d^6 M_0}{da^6} + 14b^2 c^4 \frac{d^6 M_0}{dx^2 da^4} + 14b^4 c^2 \frac{d^6 M_0}{dx^4 da^2} \right) \quad (8)
 \end{aligned}$$

Equation (7) corresponds to the equation near the top of page 346 of Weinstein's article. I have omitted the factor $n_1 n_2$, by which (7) and (8) must be multiplied if there are n_1 and n_2 turns of wire in the two coils, respectively; that is, I have so far assumed that there is only one turn of wire in each coil. The third line of equation (7), viz,

$$\frac{1}{576} \left(b_1^2 b_2^2 \frac{d^4 M_0}{d\alpha^4} + c_1^2 c_2^2 \frac{d^4 M_0}{d\alpha^4 dA^2} \right)$$

is absent altogether from Weinstein's expression, and this is the source of the error in the equation which he derived from this expression for the mutual inductance of the two coils. Weinstein does not use the sixth differentials, and hence he has only the first part of equation (8). Because of the omission of the above terms the coefficient of the third term of equation (8) is in Weinstein's expression

$$\frac{1}{960} \text{ instead of } \frac{1}{360}.$$

We may write equations (4), (5), and (6), as

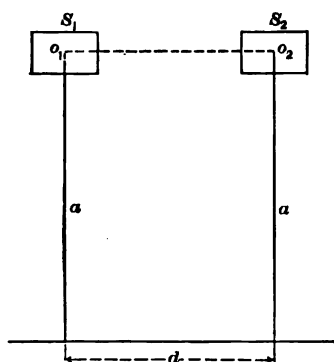


Fig. 8

$$M_1 = M_0 + \Delta_1 M$$

$$M_2 = M_1 + \Delta_2 M$$

$$\therefore M_2 = M_0 + \Delta_1 M + \Delta_2 M$$

Supposing the two coils equal, M_0 is the mutual inductance of circle O_1 and circle O_2 ; M_1 is the inductance of circle O_1 on coil S_1 or of O_2 on S_1 ; M_2 is the inductance of S_1 on S_2 . Therefore, $\Delta_1 M$ is the difference between the inductance of O_1 on S_2 and that of O_1 on O_2 ; while $\Delta_2 M$ is the difference between the inductance of coil S_1 on S_2 , and O_1 on S_2 . Obviously $\Delta_1 M$ and $\Delta_2 M$ are nearly equal, and when both are *very small* the difference can be neglected. Weinstein neglected this difference, and so took ΔM as $2\Delta_1 M$. He therefore merely doubled the value of $\Delta_1 M$ given by equation (4) first line, instead of substituting it in (5) as is done above to obtain the more accurate value given in (7) and (8). This is the source of the difference between his results and those obtained in this paper.

REVISION OF WEINSTEIN'S FORMULA.

In order to simplify the process of differentiation required in obtaining the various derivatives $\frac{d^2 M_0}{dx^2}$ etc., Weinstein transformed equation (7), not including the terms involving the sixth differentials, by means of the following relations:

$$\frac{d^2 M}{da^2} = -\frac{d^2 M}{dx^2} + \frac{1}{a} \frac{dM}{da} \quad (9)$$

$$\frac{d^3 M}{da^3} = \frac{d^3 M}{dx^3} - \frac{2}{a} \frac{d^2}{dx^2} \left(\frac{dM}{da} \right) + \frac{1}{a^2} \frac{dM}{dx^2} \quad (10)$$

$$\frac{d^4 M}{da^4 dx^2} = -\frac{d^4 M}{dx^4} + \frac{1}{a} \frac{d^3}{dx^3} \left(\frac{dM}{da} \right) \quad (11)$$

Equation (9) is Laplace's equation for cylindrical coordinates and holds for any point in space for case of symmetry about an axis. Equation (10) is obtained by differentiating (9) successively and making some substitutions, and (11) comes from (9) by differentiating it twice with respect to x .

Substituting these values in the first line of equation (8) we have for the case of two coils of equal radii and equal section:

$$\begin{aligned} \frac{M}{n_1 n_2} = M_0 + & \left\{ \frac{c^2}{12a} \frac{dM_0}{da} + \frac{1}{12} \left(b^2 - c^2 + \frac{c^4}{30a^2} \right) \frac{d^2 M_0}{dx^2} \right. \\ & \left. + \frac{c^2}{36a} \left(\frac{5b^2 - 4c^2}{20} \right) \frac{d^2 M_0}{dx^2 da} + \frac{1}{72} \left(\frac{b^4 + c^4}{5} - \frac{b^2 c^2}{2} \right) \frac{d^4 M_0}{dx^4} \dots \right\} \quad (12) \end{aligned}$$

Equation (12) corresponds to the following equation of Weinstein (at the middle of p. 349 of his article), b and c the axial breadth and radial depth of the coil corresponding to a and ρ , and x being the same as h .

$$\begin{aligned} \frac{M}{n_1 n_2} = M_0 + & \left\{ \frac{\rho^2}{12a} \frac{dM_0}{da} + \frac{1}{12} \left(a^2 - \rho^2 + \frac{\rho^4}{80a^2} \right) \frac{d^2 M_0}{dh^2} \right. \\ & \left. - \frac{\rho^2}{48a} \left(\frac{\rho^2}{10} - \frac{a^2}{3} \right) \frac{d^2 M_0}{dh^2 da} + \frac{1}{48} \left(\frac{\rho^4 + a^4}{20} - \frac{a^2 \rho^2}{3} \right) \frac{d^4 M_0}{dh^4} \dots \right\} \quad (12a) \end{aligned}$$

(The numerator of the fraction $\frac{\rho^4}{80a^3}$ is erroneously printed ρ^3 in Weinstein's paper.)

It will be seen that the coefficients in parenthesis of the several differential coefficients in equation (12) differ considerably from Weinstein's in (12a).

Maxwell's equation for the mutual inductance of two coaxial circles is

$$M = 4\pi\sqrt{Aa} \left\{ \left(\frac{2}{k} - k \right) F - \frac{2}{k} E \right\} \quad (13)$$

where

$$k = \sin \gamma = \frac{2\sqrt{Aa}}{\sqrt{(A+a)^2 + d^2}}.$$

and F and E are complete elliptic integrals to modulus k ; A and a are the radii of the circles and d is their distance apart. (In Weinstein's notation γ is λ , d is h , and F is K .)

Weinstein derived the various differential coefficients of (12a) by differentiating (13); substituting these values in (12a) he obtained an equation for M designated (1₁) page 350. M is the mutual inductance of two parallel coils of equal radii and equal section. Weinstein's formula (using his notation) is as follows:

$$\begin{aligned} M = C[K - E] & \left\{ \epsilon + \frac{\cos^2 \lambda}{12h^2} \left(a_1 - a_2 - a_3 + (2a_2 - 3a_3) \cos^2 \lambda + 8a_3 \cos^4 \lambda \right) \right\} \\ & - CE \left\{ 1 - \frac{\sin^2 \lambda}{12h^2} \left(a_1 + \frac{a_2}{2} + 2a_3 + (2a_2 + 3a_3) \cos^2 \lambda + 8a_3 \cos^4 \lambda \right) \right\} \end{aligned} \quad (14)$$

In the above formula $C = 2\pi d n_1 n_2 \sin \lambda$, where d is the mean diameter of the coils ($= 2a$), n_1 and n_2 are the number of turns of wire in the two coils respectively, and λ is the same as γ above. The mean radius being a , and the distance between the mean planes of the coils h ,

$$\sin^2 \lambda = \frac{4a^2}{4a^2 + h^2}, \quad \cos^2 \lambda = \frac{h^2}{4a^2 + h^2}$$

K and E are complete elliptic integrals (identical with F and E of 13),

a = axial breadth of the cross section of the two equal coils.

ρ = radial depth of the cross section of the two equal coils.

The four constants a_1, a_2, a_3, a_4 are defined by the following expressions:

$$\left. \begin{aligned} a_1 &= a^2 - \rho^2 + \frac{\rho^4}{20a^2} \\ a_2 &= -\frac{\rho^2}{a^2} \left(\frac{\rho^2}{10} - \frac{a^2}{3} \right) \\ a_3 &= \frac{3}{4h^2} \left(\frac{a^4 + \rho^4}{20} - \frac{a^2 \rho^2}{3} \right) \\ E &= 1 + \frac{2h^2}{a^2} + \frac{\rho^2}{6a^2} \end{aligned} \right\} \quad (15)$$

Putting the mutual inductance of a pair of coils equal to the mutual inductance of the two circles at their center of section plus a correction due to the section we have

$$M = M_0 + \Delta M$$

M_0 is then given by equation 13 (or any suitable formula), and ΔM will be the correction, plus or minus, depending on the size and shape of the section, the radius of the coils and their distance apart. If, then, we subtract (13) from (14) we shall have an expression for ΔM , which is more convenient to use in calculation than (14).

Putting

$$\left. \begin{aligned} A &= \frac{\cos^2 \lambda}{12h^2} \left(a_1 - a_2 - a_3 + (2a_1 - 3a_3) \cos^2 \lambda + 8a_3 \cos^4 \lambda \right) \\ B &= \frac{\sin^2 \lambda}{12h^2} \left(a_1 + \frac{a_2}{2} + 2a_3 + (2a_2 + 3a_3) \cos^2 \lambda + 8a_3 \cos^4 \lambda \right) \end{aligned} \right\} \quad (16)$$

we have, subtracting (13) from (14),

$$\Delta M = 4\pi a n_1 n_2 \sin \lambda \left\{ (K - E) \left(A + \frac{\rho^2}{6a^2} \right) + EB \right\} \quad (17)$$

Instead of using the values of the quantities a_1, a_2, a_3 , given by (15) we should use the values of these coefficients yielded by (12). Thus, in the notation of this article, putting a_1, a_2, a_3 , in place of a_1, a_2, a_3 , we have

$$a_1 = b^2 - c^2 + \frac{c^4}{30a^2}, \quad a_2 = \frac{5b^2c^2 - 4c^4}{60a^2}, \quad a_3 = \frac{2b^4 + 2c^4 - 5b^2c^2}{20a^2} \quad (18)$$

$$\left. \begin{aligned} A &= \frac{\cos^2 \gamma}{12a^2} \left(a_1 - a_2 - a_3 + (2a_2 - 3a_3) \cos^2 \gamma + 8a_3 \cos^4 \gamma \right) \\ B &= \frac{\sin^2 \gamma}{12a^2} \left(a_1 + \frac{a_2}{2} + 2a_3 + (2a_2 + 3a_3) \cos^2 \gamma + 8a_3 \cos^4 \gamma \right) \end{aligned} \right\} \quad (19)$$

We have now as the final expression for calculating ΔM , the correction for section,

$$\frac{\Delta M}{\pi n_1 n_2} = 4a \sin \gamma \left\{ (F - E) \left(A + \frac{c^2}{24a^2} \right) + EB \right\} \quad (20)$$

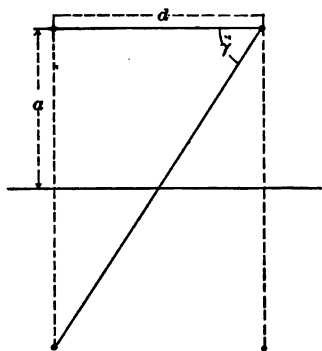


Fig. 4

In equation (20) a is as before the mean radius of the two equal coils, and γ and the other symbols have the same meaning as heretofore. Illustrations and tests of this formula will be given later in this paper.

INVESTIGATION OF STEFAN'S FORMULA.

Stefan's formula for the mutual inductance of two coaxial coils of equal radii and equal rectangular section is as follows :

$$\left. \begin{aligned} M &= 4\pi a n^2 \left\{ \log \frac{8a}{d} - 2 + \frac{b^2 - c^2}{12a^2} + \frac{2b^4 + 2c^4 - 5b^2c^2}{120a^4} + \right. \\ &\quad \left. \frac{3b^6 - 7b^4c^2 + 7b^2c^4 - 3c^6}{504a^6} + \left(\log \frac{8a}{d} - 2 \right) \left(\frac{3b^3 + c^3 + 18a^2}{96a^3} - \frac{15a^4}{1024a^4} \right) \right. \\ &\quad \left. + \frac{7b^5 + 23c^5 + 60a^2}{192a^5} - \frac{29a^4}{2048a^4} \right\} \end{aligned} \right\} \quad (21)$$

The letters, a , b , c , and d indicate as before the mean radius, the dimensions of the section and the distance apart of the mean planes of the coils.

Equation (21) can be written as follows, omitting the factor n^2 ,

$$\left. \begin{aligned} M &= 4\pi a \left\{ \log \frac{8a}{d} \left(1 + \frac{3d^2}{16a^4} + \frac{3b^2 + c^2}{96a^2} - \frac{15a^4}{1024a^4} \right) - 2 - \frac{d^2}{16a^2} + \frac{31a^4}{2048a^4} \right. \\ &\quad \left. + \frac{b^2 - c^2}{12a^2} + \frac{19c^2 - 5b^2}{192a^2} + \frac{2b^4 + 2c^4 - 5b^2c^2}{120a^4} + \frac{3b^6 - 7b^4c^2 + 7b^2c^4 - 3c^6}{504a^6} \right\} \end{aligned} \right\} \quad (22)$$

The mutual inductance of two coaxial circles of equal radii is

$$M_0 = 4\pi a \left\{ \log \frac{8a}{d} \left(1 + \frac{3a^2}{16a^2} - \frac{15a^4}{1024a^4} + \dots \right) - 2 - \frac{a^2}{16a^2} + \frac{31a^4}{2048a^4} + \dots \right\} \quad (23)$$

Subtracting (23) from (22) and putting as before ΔM for the difference between the mutual inductances of the two coils and of the two circles at their centers,

$$\Delta M = 4\pi a \left\{ \frac{3b^2 + c^2}{96a^2} \log \frac{8a}{d} + \frac{19c^2 - 5b^2}{192a^2} + \frac{b^2 - c^2}{12a^2} + \frac{2b^4 + 2c^4 - 5b^2c^2}{120a^4} + \frac{3b^6 - 7b^4c^2 + 7b^2c^4 - 3c^6}{504a^6} \right\} \quad (24)$$

This expression for ΔM derived from Stefan's equation does not (as already stated) agree either with the expression (17) derived from Weinstein's equation or with the revised expression (20).

DERIVATION OF NEW FORMULA.

To derive the formula anew we must find the values of the differential coefficients of equation (8), by differentiating the series formula for the mutual inductance of two coaxial circles of unequal radii. Substituting in (8) will then give an expression for ΔM . Weinstein did this as we have seen by using the expression (13) in elliptic integrals. Stefan must have used the series formula, differentiating up to the sixth order for the principal terms. The mutual inductance of two coaxial circles of radii a and $a+y$ is given by the following series:⁷

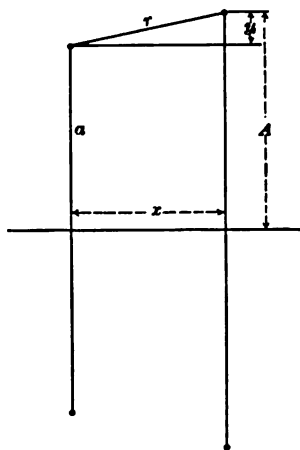


Fig. 5

$$M = 4\pi a \left\{ \log \frac{8a}{r} \left(1 + \frac{y}{2a} + \frac{3x^2 + y^2}{16a^2} - \frac{3x^2y + y^3}{32a^3} - \frac{15x^4 - 42x^2y^2 - 17y^4}{1024a^4} \right. \right. \\ \left. + \frac{45x^4y - 30x^2y^3 - 19y^5}{2048a^5} + \dots \right) - \left(2 + \frac{y}{2a} + \frac{x^2 - 3y^2}{16a^2} - \frac{6x^2y - y^3}{48a^3} \right. \\ \left. - \frac{93x^4 - 534x^2y^2 - 19y^4}{6144a^4} + \frac{1845x^4y - 3030x^2y^3 - 379y^5}{61440a^5} + \dots \right) \right\} \quad (25)$$

⁷ Rosa and Cohen. This Bulletin, p. 366.

where a is the radius of the smaller circle, $A = a + y$, is the radius of the larger circle, x is the distance between the planes of the circles.

$$r = \sqrt{x^2 + y^2}$$

To differentiate the entire equation at once for all the coefficients of (8) would be confusing. We may better divide it into parts and differentiate them separately. Put

$$M' = 4\pi a \left\{ \left(1 + \frac{y}{2a} \right) \log \frac{8a}{r} - 2 - \frac{y}{2a} \right\}$$

$$M'' = 4\pi a \left\{ \frac{3x^2 + y^2}{16a^2} \log \frac{8a}{r} - \frac{x^2 - 3y^2}{16a^2} \right\}$$

$$M''' = -4\pi a \left\{ \frac{3x^2 y + y^3}{32a^3} \log \frac{8a}{r} - \frac{6x^2 y - y^3}{48a^3} \right\}$$

$$M'''' = -4\pi a \left\{ \frac{15x^4 - 42x^2 y^2 - 17y^4}{1024a^4} \log \frac{8a}{r} - \frac{93x^4 - 534x^2 y^2 - 19y^4}{6144a^4} \right\}$$

$$M''''' = 4\pi a \left\{ \frac{45x^4 y - 30x^2 y^3 - 19y^5}{2048a^5} \log \frac{8a}{r} - \frac{1845x^4 y - 3030x^2 y^3 - 379y^5}{61440a^5} \right\}$$

$$\text{Then } M = M' + M'' + M''' + M'''' + M'''''$$

Evidently M' is the principal part of M when the two circles are of nearly equal radii and near each other. As the sections of the two coils of which we are to determine the mutual inductance are supposed to be small in comparison with their radii, and the radii are to be equal, y will always be relatively small. When the coils are close together x will also be small. In the latter case, M' is therefore nearly the whole of M , and the sixth differentials required will depend almost entirely on M' . On the other hand, when the coils are farther apart M'' is important, and we may expect an important part of the correction to depend on it, and to some extent even on M''' and M'''' , but it will not be necessary to differentiate these latter expressions as far as the first.

To differentiate M' we may write it

$$\frac{M'}{2\pi} = (A + a) \log \frac{8a}{\sqrt{(A - a)^2 + x^2}} - A - 3a \quad (26)$$

M' must be differentiated successively with respect to A , a , and x .

$$\begin{aligned} \frac{1}{2\pi} \frac{dM'}{da} &= \log \frac{8a}{\sqrt{(A-a)^2 + x^2}} + \frac{A^2 - a^2}{(A-a)^2 + x^2} + \frac{A}{a} - 2 \quad (27) \\ \frac{1}{2\pi} \frac{d^2 M'}{da^2} &= \frac{A-a}{(A-a)^2 + x^2} + \frac{1}{a} - \frac{2a}{(A-a)^2 + x^2} + \frac{2(A+a)(A-a)^2}{[(A-a)^2 + x^2]^2} - \frac{A}{a^2} \end{aligned}$$

when $y = A - a = 0$, and $x = d$,

$$\frac{d^2 M'_0}{da^2} = 4\pi a \left[-\frac{1}{d^2} \right] \quad (28)$$

This is the value of the first differential coefficient occurring in (7) and (8) so far as it depends on M' .

Differentiating M'' , M''' , and M'''' in a similar manner, and putting $A = a$ and $y = d$, we get for the sum of the four differential coefficients the following, omitting terms of higher power in $1/a$ than $1/a^4$.

$$-\frac{d^2 M'_0}{da^2} = 4\pi a \left\{ -\frac{1}{d^2} + \frac{3}{16a^2} + \frac{1}{8a^3} \log \frac{8a}{d} + \frac{42a^2}{512a^4} \log \frac{8a}{d} - \frac{163a^2}{1024a^4} \right\} \quad (29)$$

$$\frac{d^2 M'_0}{dx^2} = 4\pi a \left\{ +\frac{1}{d^2} - \frac{11}{16a^2} + \frac{3}{8a^3} \log \frac{8a}{d} - \frac{90a^2}{512a^4} \log \frac{8a}{d} + \frac{291a^2}{1024a^4} \right\} \quad (30)$$

From (27) when $A = a$ and $x = d$ we have

$$\frac{1}{a} \frac{dM'_0}{da} = 4\pi a \left\{ \frac{1}{2a^2} \log \frac{8a}{d} - \frac{1}{2a^3} \right\} \quad (31)$$

and if we take the same derivatives from M'' , and M''' , and M'''' and add the four values we obtain the following, omitting as before terms of higher than the fourth degree in $1/a$:

$$\frac{1}{a} \frac{dM'_0}{da} = 4\pi a \left\{ \frac{1}{2a^2} \log \frac{8a}{d} - \frac{1}{2a^3} + \frac{a^2}{8a^4} - \frac{3a^2}{32a^4} \log \frac{8a}{d} \right\} \quad (32)$$

Equation (9) should hold in this case and give us a check on the work of differentiation, as well as a check upon the correctness of formula (25). If we have not differentiated enough terms of (25) to get all the terms of degrees contained in 29, 30, and 32 equation (9) would not be satisfied.

Equation (9) is

$$\frac{d^2 M}{da^2} + \frac{d^2 M}{dx^2} = \frac{1}{a} \frac{dM}{da} \quad (33)$$

The sum of the right hand members of (29) and (30) is

$$4\pi a \left\{ \frac{1}{2a^3} \log \frac{8a}{d} - \frac{8}{16a^3} + \frac{128a^3}{1024a^4} - \frac{48a^3}{512a^4} \log \frac{8a}{d} \right\} \quad (34)$$

which is identically equal to (32), thus satisfying (9).

Proceeding now to the fourth differentials we find the following as the principal terms:

$$\left. \begin{aligned} \frac{d^4 M_0}{da^4} &= \frac{d^4 M_0}{dA^4} = \frac{d^4 M_0}{dx^4} = 4\pi a \left[+\frac{6}{d^4} \right] \\ \frac{d^4 M_0}{da^3 dx} &= \frac{d^4 M^2}{dA^3 dx} = 4\pi a \left[-\frac{6}{d^4} \right] \end{aligned} \right\} \quad (35)$$

The expressions become complicated for these higher differentials, but we may simplify the process by transforming the equation for M . Thus, putting y for $A-a$ in equation (26), remembering that $\frac{dy}{dA} = 1$,

$$\begin{aligned} \frac{M'}{2\pi} &= (2a+y) \log 8a - \left(a + \frac{y}{2}\right) \log (x^2 + y^2) - (4a+y) \\ \text{Or, } \frac{M'}{2\pi} &= (2a+y) \log 8a - (4a+y) - \left(a + \frac{y}{2}\right) \log x^2 \\ &\quad - \left(a + \frac{y}{2}\right) \log \left(1 + \frac{y^2}{x^2}\right) \\ &= (2a+y) \log 8a - (4a+y) - (2a+y) \log x \\ &\quad - \left(a + \frac{y}{2}\right) \left[\frac{y^2}{x^2} - \frac{1}{2} \frac{y^4}{x^4} + \frac{1}{3} \frac{y^6}{x^6} - \dots \right] \\ \therefore \frac{1}{2\pi} \frac{dM'}{dA} &= \log 8a - 1 - \log x - a \left[\frac{2y}{x^3} - \frac{2y^3}{x^5} + \frac{2y^5}{x^7} - \dots \right] \\ &\quad - \frac{1}{2} \left[\frac{3y^3}{x^3} - \frac{5y^5}{2x^5} + \frac{7y^7}{3x^7} - \dots \right] \\ \frac{1}{2\pi} \frac{d^2 M'}{dA^2} &= -a \left[\frac{2}{x^3} - \frac{6y^2}{x^5} + \frac{10y^4}{x^7} - \dots \right] - \frac{1}{2} \left[\frac{6y}{x^3} - \frac{10y^3}{x^5} + \frac{14y^5}{x^7} - \dots \right] \\ \text{If } y=0 \text{ and } x=d, \frac{d^2 M_0'}{dA^2} &= 4\pi a \left[-\frac{1}{d^3} \right] \text{ as found previously.} \end{aligned} \quad (36)$$

Proceeding with the differentiation

$$\frac{1}{2\pi} \frac{d^3 M'}{dA^3} = -a \left[-\frac{12y}{x^4} + \frac{40y^3}{x^6} - \dots \right] - \frac{1}{2} \left[\frac{6}{x^2} - \frac{30y^2}{x^4} + \frac{70y^4}{x^6} - \dots \right]$$

$$\frac{1}{2\pi} \frac{d^3 M'}{dA^3} = -a \left[-\frac{12}{x^4} + \frac{120y^2}{x^6} - \dots \right] - \frac{1}{2} \left[-\frac{60y}{x^4} + \frac{280y^3}{x^6} - \dots \right]$$

$$\therefore \frac{d^3 M'_0}{dA^3} = 4\pi a \left[+\frac{6}{a^3} \right], \text{ as found previously.}$$

$$\frac{1}{2\pi} \frac{d^3 M'}{dA^3} = -a \left[\frac{240y}{x^6} - \dots \right] - \frac{1}{2} \left[-\frac{60}{x^4} + \frac{840y^2}{x^6} - \dots \right]$$

$$\frac{1}{2\pi} \frac{d^3 M'}{dA^3} = -a \left[\frac{240}{x^6} - \dots \right] - \frac{1}{2} \left[\frac{1680y}{x^6} - \dots \right]$$

All terms in the series that have been omitted have y as a factor, and hence when $y=0$ they all vanish. Hence there is only one term in the entire series that does not vanish for $\frac{d^3 M'_0}{dA^3}$. Thus we have

$$\frac{d^3 M'_0}{dA^3} = 4\pi a \left[-\frac{120}{a^3} \right]$$

The same method is successful in obtaining the differential coefficients with respect to a and x . The following are the values:

$$\left. \begin{aligned} \frac{d^3 M'_0}{da^3} &= \frac{d^3 M'_0}{dA^3} = \frac{d^3 M'_0}{dx^4 da^3} = 4\pi a \left[-\frac{120}{a^3} \right] \\ \frac{d^3 M'_0}{dx^6} &= \frac{d^3 M'_0}{dx^2 da^4} = 4\pi a \left[+\frac{120}{a^3} \right] \end{aligned} \right\} (37)$$

We may safely neglect the fourth and sixth differentials of M'' and M''' , which latter are very small where the field is changing rapidly, that is, near the primary coil.

Substituting now the values of the coefficients as given by (29), (30), (35), and (37) in equation (8), in which the coils are assumed

to have the same radii and section, we obtain the following expression for ΔM :

$$\begin{aligned} \Delta M = 4\pi a \left\{ \frac{3b^3 + c^3}{96a^3} \log \frac{8a}{d} - \frac{11b^3 - 3c^3}{192a^3} + \frac{b^3 - c^3}{12a^3} + \frac{2b^4 + 2c^4 - 5b^3c}{120a^3} \right. \\ \left. + \frac{3b^5 - 3c^5 + 14b^3c^2 - 14b^4c}{504d^3} + \frac{7c^3d^2}{1024a^4} \left(\log \frac{8a}{d} - \frac{163}{84} \right) \right. \\ \left. - \frac{15b^3d^2}{1024a^4} \left(\log \frac{8a}{d} - \frac{97}{60} \right) \right\} \quad (38) \end{aligned}$$

The second and fifth terms of the equation (38) differ from the corresponding terms of (24) which was derived from Stefan's formula, the principal difference being in the second term, and there are two additional terms in (38). For a square section the second term is negative in (38) and positive and larger in (24). It is the error in this term chiefly which makes Stefan's formula give too large values. This is especially noticeable when the coils are far apart, when the other corrections are small. When the coils are near together the other corrections are so much larger that this error is obscured, and the formula appears more nearly correct.

Instead of substituting in (8) we might have substituted in (12) the transformed equation used by Weinstein, adding the second line of (8) to (12) to give the terms depending on the sixth differentials. This requires the value of one other differential coefficient, viz:

$$\frac{1}{a} \frac{d^3 M_0}{dx^2 da} = 4\pi a \left\{ \frac{1}{2a^3 d^3} - \frac{3}{16a^4} \log \frac{8a}{d} \right\}$$

Carrying out these substitutions we obtain the same expression (38) with some additional very small terms, which are negligible, except for coils of very large section, or coils very near together, in which case one term is appreciable. This term is—

$$\Delta_2 M = 4\pi a \left\{ \frac{6b^4 + 6c^4 + 5b^3c^3}{5760a^3 d^3} \right\} \quad (39)$$

and may be used if needed by adding it to (38). It is included in (41) below. For a square section, where $b=c$, the third and fifth terms of (38) disappear and the formula becomes; adding (39):

$$\Delta M = 4\pi a \left\{ \frac{b^3}{24a^3} \left(\log \frac{8a}{d} - 1 \right) - \frac{b^3 d^3}{128a^4} \left(\log \frac{8a}{d} - \frac{4}{3} \right) - \frac{b^4}{120d^4} + \frac{17b^4}{5760a^3 d^3} \right\} \quad (40)$$

$$\text{Or, } \Delta M = \frac{\pi b^3}{6a} \left\{ \log \frac{8a}{d} - 1 - \frac{3d^3}{16a^3} \left(\log \frac{8a}{d} - \frac{4}{3} \right) - \frac{a^3 b^3}{5d^4} + \frac{17b^3}{240ad^3} \right\} \quad (41)$$

The following approximate formula for coils of square section is sufficiently exact for most purposes, although varying more from the exact expression as d is greater:

$$\Delta M = \frac{\pi b^3}{6a} \left\{ \log \frac{8a}{d} - 1 - \frac{a^3 b^3}{5d^4} \right\} \quad (42)$$

When the two coils have equal radii but unequal sections substitution must be made in (7) instead of (8). Neglecting sixth differentials, which are inappreciable except for coils very near, we obtain for coils not very far apart the following expression:

$$\begin{aligned} \Delta M = 4\pi a \left\{ \frac{3(b_1^3 + b_2^3) + c_1^3 + c_2^3}{192a^3} \log \frac{8a}{d} - \frac{11(b_1^3 + b_2^3) - 3(c_1^3 + c_2^3)}{384a^3} \right. \\ \left. + \frac{(b_1^3 + b_2^3) - (c_1^3 + c_2^3)}{24a^3} \right. \\ \left. + \frac{(3b_1^4 + 10b_1^2 b_2^2 + 3b_2^4) + (3c_1^4 + 10c_1^2 c_2^2 + 3c_2^4) - 10(b_1^3 + b_2^3)(c_1^3 + c_2^3)}{960a^4} \right\} \quad (43) \end{aligned}$$

In every case the value of ΔM is to be multiplied by $n_1 n_2$, where n_1 and n_2 are the number of turns of wire on the two coils.

COILS OF A SINGLE LAYER. CURRENT SHEETS.

By putting $c=0$ in formula (38) we get an expression for ΔM for two coaxial current sheets, which would be realized substantially by single layer coils, so far as mutual inductance is concerned. If we consider their radii as fixed, and therefore a constant, equations (4) and (5) would be simplified by the disappearance of all terms in-

volving differential coefficients with respect to a . Equation (7) would then become:

$$\begin{aligned}
 M = M_0 &+ \frac{1}{24} (b_1^2 + b_2^2) \frac{d^2 M_0}{dx^2} \\
 &+ \left(\frac{b_1^4 + b_2^4}{1920} + \frac{b_1^2 b_2^2}{576} \right) \frac{d^4 M_0}{dx^4} \\
 &+ \left(\frac{b_1^6 + b_2^6}{120 \cdot 2688} + \frac{b_1^4 b_2^2 + b_1^2 b_2^4}{24 \cdot 1920} \right) \frac{d^6 M_0}{dx^6} \\
 &+ \left(\frac{b_1^8 + b_2^8}{120 \cdot 256 \cdot 3024} + \frac{b_1^6 b_2^2 + b_1^2 b_2^6}{24 \cdot 120 \cdot 2688} + \frac{b_1^4 b_2^4}{(1920)^2} \right) \frac{d^8 M_0}{dx^8} + \dots \quad (44)
 \end{aligned}$$

When the coils have equal lengths, i. e., $b_1 = b_2 = b$,

$$M = M_0 + \frac{b^2}{12} \frac{d^2 M_0}{dx^2} + \frac{b^4}{360} \frac{d^4 M_0}{dx^4} + \frac{b^6}{120 \cdot 168} \frac{d^6 M_0}{dx^6} + \frac{b^8}{864 \cdot 2100} \frac{d^8 M_0}{dx^8} + \dots \quad (45)$$

Taking Coffin's extension of Maxwell's equation for the mutual inductance of two coaxial circles of equal radius we can easily obtain the differential coefficients to the eighth order, to substitute in (45). The expression for M , for the two equal circles is as follows:

$$\begin{aligned}
 M = 4\pi a \left\{ \log \frac{8a}{d} \cdot \left(1 + \frac{3x^2}{16a^2} - \frac{15x^4}{8 \cdot 128a^4} + \frac{35x^6}{128^2 a^6} - \frac{1575x^8}{2 \cdot 128^3 a^8} + \dots \right) \right. \\
 \left. - \left(2 + \frac{x^2}{16a^2} - \frac{31x^4}{16 \cdot 128a^4} + \frac{247x^6}{6 \cdot 128^2 a^6} - \frac{7,795x^8}{8 \cdot 128^3 a^8} + \dots \right) \right\} \quad (46)
 \end{aligned}$$

Because y is zero, the equation is simpler to the eighth degree than equation (25) to the fifth degree, and the differentiation is greatly simplified by making a constant.

The values of the differential coefficients are as follows:

$$\frac{d^2 M_0}{dx^2} = 4\pi a \left\{ \log \frac{8a}{d} \cdot \left(\frac{3}{8a^2} - \frac{45d^2}{256a^4} + \frac{1050d^4}{128^2 a^6} - \frac{44100d^6}{128^3 a^8} \dots \right) \right\} \quad (47)$$

$$\frac{d^4 M_0}{dx^4} = 4\pi a \left\{ \frac{3}{8a^2 d^2} + \frac{6}{d^4} - \frac{45}{128a^4} \left(\log \frac{8a}{d} - \frac{187}{60} \right) \right\} \quad (48)$$

$$\frac{d^6 M_0}{dx^6} = 4\pi a \left\{ \frac{120}{d^6} \right\} \quad (49)$$

$$\frac{d^3 M_0}{dx^3} = 4\pi a \left\{ \frac{5040}{d^3} \right\} \quad (50)$$

Substituting these values of the differential coefficients in (45) we get for ΔM the following expression:

$$\begin{aligned} \Delta M = 4\pi a \left[\frac{b^3}{12d^3} + \frac{b^3}{32a^3} \left(\log \frac{8a}{d} - \frac{11}{6} \right) - \frac{15b^3 d^3}{1024a^4} \left(\log \frac{8a}{d} - \frac{97}{60} \right) \right. \\ \left. + \frac{175b^3 d^4}{2(128)^2 a^5} \left(\log \frac{8a}{d} - \frac{54}{35} \right) - \frac{3675b^3 d^4}{(128)^3 a^6} \left(\log \frac{8a}{d} - \frac{3793}{2520} \right) + \frac{b^4}{960a^3 d^3} \right. \\ \left. + \frac{b^4}{60d^4} - \frac{b^4}{1024a^4} \left(\log \frac{8a}{d} - \frac{187}{60} \right) + \frac{b^4}{168d^6} + \frac{b^8}{360d^8} \right] \quad (51) \end{aligned}$$

This formula will give the mutual inductance (adding ΔM to M_0) with very great precision for two coaxial single layer coils of equal radii, provided the coils are not in contact or d is not too great. If d is equal to or greater than a we can obtain a more accurate value by using formula (20) making $c=0$. In this case

$$\begin{aligned} A &= \frac{b^3 \cos^2 \gamma}{12d^3} \left(1 - \frac{b^2}{10d^2} - \frac{3b^2}{10d^2} \cos^2 \gamma + \frac{4b^2}{5d^2} \cos^4 \gamma \right) \\ B &= -\frac{b^3 \sin^2 \gamma}{12d^3} \left(1 + \frac{b^2}{5d^2} + \frac{3b^2}{10d^2} \cos^2 \gamma + \frac{4b^2}{5d^2} \cos^4 \gamma \right) \\ \Delta M &= 4\pi a \sin \gamma \{ (F-E) A + EB \} \quad (52) \end{aligned}$$

If the coils are considerably distant from one another this expression is very exact. If quite near we may improve the accuracy by adding to the value of ΔM obtained by (52) the last two terms of (51) which represent the part of ΔM depending on sixth and eighth differentials. Examples of the use of these formulæ on single layer coils will be found in another paper in this bulletin.*

We must now test the formulæ for ΔM by numerical applications to see how closely it approximates to the true value of the correction ΔM .

* Rosa and Cohen, this Bulletin, p. 408.

TESTS OF FORMULÆ 38 AND 41.

Let there be two coaxial coils of mean radii 25 cm, and square section 2 x 2 cm, and distant 4 cm between their mean planes. Thus

$$a = 25$$

$$b = 2$$

$$c = 2$$

$$d = 4$$

$$\log_e \frac{8a}{d} = \log_e 50 = 3.9120$$

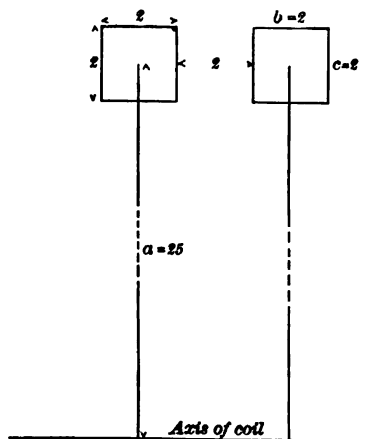


Fig. 6

Substituting these values in (41) we have

$$\log \frac{8a}{d} - 1 = 2.9120$$

$$\frac{17b^3}{240d^3} = \frac{.0177}{150} = 2.9297$$

$$-\frac{a^3b^3}{5d^4} = -1.9531$$

$$-\frac{3d^3}{16a^3} \left(\log \frac{8a}{d} - \frac{4}{3} \right) = \frac{.0124}{150} = -\frac{1.9655}{+0.9642}$$

$$\frac{b^3}{6a} = \frac{4}{150}; \quad 0.9642 \times \frac{4}{150} = .02571$$

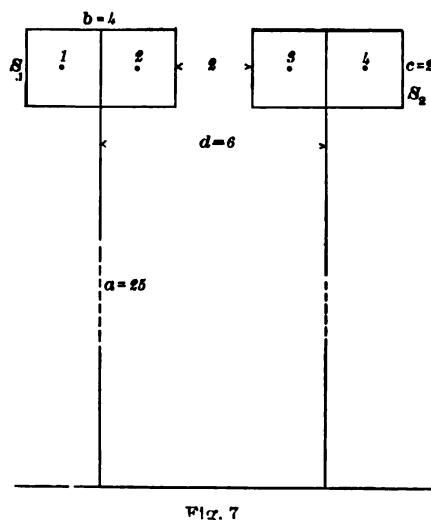
$$\therefore \frac{\Delta M}{\pi} = +0.02571 \quad (53)$$

The value of $\frac{M_0}{\pi}$ as given by (13) or (23) is 192.9174, assuming only

one turn in each coil. The correction ΔM is therefore about one part in 7,500. If there were 400 turns of wire in each coil the mutual inductance would be

$$\begin{aligned} M &= 400^2 \pi (192.9174 + .0257) \\ &= 96,983,800 \text{ cm} \\ &= 96.9838 \text{ millihenrys.} \end{aligned}$$

If the coils be moved apart farther the correction ΔM increases, and at $d=6$ cm, $\frac{\Delta M}{\pi} = .05613$ more than double its value at 4 cm. This is contrary to what one would at first expect, until one remem-



bers that for coils close together the correction is negative. At $d=3$ cm it is still negative, while at some point between 3 and 4 it is zero. At $d=8$ cm the correction $\frac{\Delta M}{\pi}$ is .05373 by formula (41).

These corrections are all relatively small, and have been computed with great accuracy, giving the mutual inductances M at these various distances with high precision.

Let there be two coils S_1 and S_2 each of rectangular section 2×4 cm, and their mean planes distant 6 cm. Each coil may be regarded as made up of two coils of square section 2×2 . The mutual induc-

tance of the two coils S_1 and S_2 upon one another is made up of the sum of the mutual inductances of coils 1 and 2 upon coils 3 and 4, coils 1 and 2 being the two halves of S_1 and 3 and 4 the two halves of S_2 . Thus,

$$M = M_{13} + M_{14} + M_{23} + M_{24}$$

M_{13} is of course equal to M_{24} .

We have given above the values of $\frac{\Delta M}{\pi}$ for each of these separate cases and we can now obtain the mutual inductance of S_1 on S_2 . Thus

	$\frac{M_0}{\pi}$	$\frac{\Delta M}{\pi}$
M_{13}	154.0718	.05613
M_{14}	127.3947	.05373
M_{23}	192.9174	.02571
M_{24}	154.0718	.05613
$\Sigma (M_0 \div \pi)$	628.4557	.19170
$\Sigma (\Delta M \div \pi)$	.1917	
$\frac{M}{\pi} =$	628.6474	

We can now determine the value of M by another process. Regarding the coils S_1 and S_2 as made up of two parts of one turn each, calculate M_0 by (13) or (23) and ΔM by (38). The distance apart of the mean planes being 6 cm, we see that M_0 is four times M_{13} , that is,

$$\text{For } S_1 \text{ on } S_2, \frac{M_0}{\pi} = 616.2872$$

To calculate ΔM , substitute in (38) the following values:

$$\begin{aligned} a &= 25 \\ b &= 4 \\ c &= 2 \\ d &= 6 \end{aligned} \quad \log_e \frac{8a}{d} = \log_e \frac{200}{6} = 3.5065$$

$$\begin{aligned}
\frac{3b^3+c^3}{96a^3} \log \frac{8a}{d} &= .003038 \\
\frac{b^3-c^3}{12d^3} &= .027778 \\
\frac{2b^4+2c^4-5b^3c^3}{120d^4} &= .001440 \\
\frac{3b^5-3c^5+14b^3c^4-14b^4c^3}{504d^5} &= .000057 \\
\frac{6b^6+6c^6+5b^3c^3}{5760a^3d^3} &= .000015 \\
\frac{7c^3d^3}{1024a^4} \left(\log \frac{8a}{d} - \frac{163}{84} \right) &= .000004 = .032332 \\
-\frac{15b^3-3c^3}{192a^3} &= -.001367 \\
-\frac{15b^3d^3}{1024a^4} \left(\log \frac{8a}{d} - \frac{97}{60} \right) &= .000040 = -.001407 \\
& .030925
\end{aligned}$$

$$4an^3 = 400 \quad \therefore \frac{4M}{\pi} = 12.370$$

$$\frac{M_0}{\pi} = 616.287$$

$$\therefore \frac{M}{\pi} = 628.657$$

$$\text{By process of summation above } M/\pi = 628.647$$

$$\text{Difference} = .010$$

which is about one part in 60,000. The correction by the process of summation is so small that we may be sure it is more exact than the second method, in which the correction is more than 2 per cent. To give absolute agreement between the results of the two methods the second correction should be 12.360 instead of 12.370, a difference of less than one-tenth of one per cent of the correction. This is a very good test of the formula (38) for this distance between the coils, and the accuracy is very satisfactory.

TEST OF FORMULA 20.

Let formula (20) be applied to the same coil to which 41 has been applied, where

$$\begin{array}{ll} a = 25 & \sin^2 \gamma = \frac{2500}{2516} \\ b = 2 & \\ c = 2 & \cos^2 \gamma = \frac{16}{2516} \\ d = 4 & \end{array}$$

Substituting the above values of the constants in (18) we find the following values of a_1, a_2, a_3 :

$$\begin{array}{ll} a_1 = & .000853 \\ a_2 = & .000427 \\ a_3 = & -.050000 \\ \frac{c^2}{24a^3} = & .0002667 \end{array}$$

Substituting in (19) we obtain

$$\begin{array}{ll} A + \frac{c^2}{24a^3} = & .0002684 \\ B = & -.0005170 \end{array}$$

The complete elliptic integrals to modulus $\sin \gamma$ have the following values for this particular case:

$$\begin{array}{ll} F & = 3.91986 \\ E & = 1.01088 \\ F-E & = 2.90898 \\ \therefore (F-E) \left(A + \frac{c^2}{24a^3} \right) & = .00078077 \\ EB & = -.00052262 \\ \text{Sum} & = .00025815 \\ 4a \sin \gamma & = 99.6815 \\ \therefore \frac{AM}{\pi} & = .02573 \end{array}$$

Formula (41) gave .02571, which is almost in absolute agreement, the last figure being dropped in the final calculations. The uncertainty in this value is probably not greater than one in a million of the whole inductance of these two coils.

Taking now the larger coils S_1 and S_{11} , Fig. (7), where $b=4$, and substituting in equation (20) we get the larger correction $\frac{\Delta M}{\pi}$. Following the process indicated above we find for S_1 and S_{11} ,

$$\frac{\Delta M}{\pi} = 12.349$$

$$\text{As before, } \frac{M_0}{\pi} = \frac{616.287}{\pi}$$

$$\therefore M = 628.636$$

$$\text{Value by summation, } \frac{628.647}{\pi}$$

$$\text{Difference} = .011 = 1 \text{ part in } 57,000.$$

This is nearly as good agreement as by formulae (38) and (41).

There are decided advantages in calculating the quantity ΔM separately and adding it to M_0 to obtain M , the mutual inductance of two coils. For M_0 can be calculated by more than one process, as a check on the numerical work as well as a check on the formulæ, and similarly ΔM can be calculated by more than one formula. When they are put together in one formula one can not be sure to what any difference found between the results of two different formulæ is due. There is no trouble in computing M_0 to any desired degree of accuracy. Practically the whole problem is to find ΔM . Hence it is safer as well as more convenient when practicable to have separate formulæ for ΔM .

The above tests of these formulæ show that for coils of this size and distance apart very accurate values of the mutual inductance can be obtained, as accurate as will be required in the most exact experimental work. Of course, if the cross sections are greater the accuracy will be less, but in work of precision large cross sections should not be used; the depth would probably seldom be greater than 2 cm. Formulæ (38) and (41) give values for ΔM a little less exact as the distance apart of the coils increases, since the terms neglected in the series (25) becomes appreciable when d is large. For such cases formula (20) is more exact. Further tests and discussion of these formulæ are given in another paper⁹ in this Bulletin.

⁹ Rosa and Cohen, this Bulletin, p. 359.

THE MUTUAL INDUCTANCE OF TWO CIRCULAR COAXIAL COILS OF RECTANGULAR SECTION.

By Edward B. Rosa and Louis Cohen.

Various formulæ have been proposed from time to time for calculating the mutual inductance of coaxial coils. All of them are approximate formulæ and in most cases the approximation is closer if the coils are not near each other. The degree of approximation is not, however, shown by the formulæ themselves, and it is therefore highly important to critically examine and compare all the formulæ available, and to ascertain which are most accurate and what the magnitude of the residual error is likely to be in any given case. A practical question, for example, is to determine what limitations as to size of section, radius, and distance apart must be placed on two coils in order that their mutual inductances may be computed to one part in 50,000. If such coils are to be used in the absolute measurement of resistance, the formulæ employed for computing the mutual inductance must be justified beyond question. We propose in this paper to examine these formulæ and to compare them by numerous numerical calculations. We shall show which are the more accurate formulæ, point out where some of them fail, and shall derive some new expressions more convenient to use than some of those which have heretofore been employed. We shall also give a number of examples to illustrate and test the formulæ, and curves to show the relative accuracy of various formulæ for particular coils at varying distances.

MUTUAL INDUCTANCE OF TWO COAXIAL CIRCLES.

MAXWELL'S FORMULÆ IN ELLIPTIC INTEGRALS.

Some of the formulæ available give the mutual inductance of two coaxial coils directly in terms of the dimensions of the coils, while others derive it from the mutual inductance of two coaxial circles, either by giving the correction to be applied to the latter, or by employing in the formula for the latter a modified radius, or by combining several values for circles in such a way as to give (approximately) the value for coils of appreciable cross section. It is therefore desirable to consider first the formulæ for the calculation of the mutual inductance of two coaxial circles. The first and most important is the formula in elliptic integrals given by Maxwell:¹

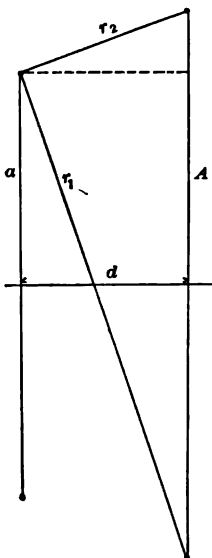


Fig. 1

$$M = 4\pi\sqrt{Aa} \left\{ \left(\frac{2}{k} - k \right) F - \frac{2}{k} E \right\} \quad (1)$$

in which A and a are the radii of the two circles, d is the distance between their centers, and

$$k = \frac{2\sqrt{Aa}}{\sqrt{(A+a)^2 + d^2}} = \sin \gamma$$

F and E are the complete elliptic integrals of the first and second kind, respectively, to modulus k . Their values may be obtained from the tables of Legendre, or the values of $M \div 4\pi\sqrt{Aa}$ may be obtained from the table in Appendix I at the end of Chapter XIV (Vol. II) of Maxwell, the values of γ being the argument.

The notation of Maxwell is slightly altered in the above expressions in order to bring it into conformity with the formulæ to follow.

Formula (1) is an absolute one, giving the mutual inductance of two coaxial circles of any size at any distance apart. If the two circles have equal or nearly equal radii, and are very near each other, the quantity k will be very nearly equal to unity and γ will be near

¹ Electricity and Magnetism, Vol. II, § 701.

to 90° . Under these circumstances it may be difficult to obtain a sufficiently exact value of F and E from the tables, as the quantities are varying rapidly and it is necessary to employ an interpolation formula to get values between those given in the tables. Under such circumstances the following formula, also given by Maxwell (derived by means of Landen's transformation), is more suitable:

$$M = 8\pi \sqrt{\frac{Aa}{k_1}} \left\{ F_1 - E_1 \right\} \quad (2)$$

in which F_1 and E_1 are complete elliptic integrals to modulus k_1 , and

$$k_1 = \frac{r_1 - r_2}{r_1 + r_2} = \sin \gamma_1$$

r_1 and r_2 are the greatest and least distances of one circle from the other (Fig. 1); that is,

$$r_1 = \sqrt{(A+a)^2 + d^2}$$

$$r_2 = \sqrt{(A-a)^2 + d^2}$$

The new modulus k_1 differs from unity more than k , hence γ_1 is not so near to 90° as γ and the values of the elliptic integrals can be taken more easily from the tables than when using formula (1) and the modulus k .

Another way of avoiding the difficulty when k is nearly unity is to calculate the integrals F and E directly, and thus not use the tables of elliptic integrals, expanding F and E in terms of the complementary modulus k' , where $k' = \sqrt{1 - k^2}$. The expressions for F and E are very convergent when k' is small. For convenience of reference they are here given. An example will be given later to illustrate the use of these formulæ.

$$\begin{aligned} F = & \log \frac{4}{k'} + \frac{1^2}{2^2} k'^2 \left(\log \frac{4}{k'} - \frac{2}{1.2} \right) \\ & + \frac{1^2}{2^2} \frac{3^2}{4^2} k'^4 \left(\log \frac{4}{k'} - \frac{2}{1.2} - \frac{2}{3.4} \right) \\ & + \frac{1^2}{2^2} \frac{3^2}{4^2} \frac{5^2}{6^2} k'^6 \left(\log \frac{4}{k'} - \frac{2}{1.2} - \frac{2}{3.4} - \frac{2}{5.6} \right) \\ & + \frac{1^2}{2^2} \frac{3^2}{4^2} \frac{5^2}{6^2} \frac{7^2}{8^2} k'^8 \left(\log \frac{4}{k'} - \frac{2}{1.2} - \frac{2}{3.4} - \frac{2}{5.6} - \frac{2}{7.8} \right) \\ & + \dots \dots \dots \end{aligned} \quad (3)$$

$$\begin{aligned}
 E = & 1 + \frac{1}{2}k'^2 \left(\log \frac{4}{k'} - \frac{1}{1.2} \right) \\
 & + \frac{1^2}{2^2} \frac{3}{4} k'^4 \left(\log \frac{4}{k'} - \frac{2}{1.2} - \frac{1}{3.4} \right) \\
 & + \frac{1^2}{2^2} \frac{3^2}{4^2} \frac{5}{6} k'^6 \left(\log \frac{4}{k'} - \frac{2}{1.2} - \frac{2}{3.4} - \frac{1}{5.6} \right) \\
 & + \frac{1^2}{2^2} \frac{3^2}{4^2} \frac{5^2}{6^2} \frac{7}{8} k'^8 \left(\log \frac{4}{k'} - \frac{2}{1.2} - \frac{2}{3.4} - \frac{2}{5.6} - \frac{1}{7.8} \right) \\
 & + \dots \dots \dots
 \end{aligned} \tag{4}$$

WEINSTEIN'S FORMULA.

Weinstein² gives an expression for the mutual inductance of two coaxial circles, in terms of the complementary modulus k' used in the preceding series (3) and (4). That is, substituting the values of F and E given above in equation (1) we have Weinstein's equation, which is as follows:

$$\begin{aligned}
 M = 4\pi\sqrt{Aa} \left\{ \left(1 + \frac{3}{4}k'^2 + \frac{33}{64}k'^4 + \frac{107}{256}k'^6 + \frac{5913}{16384}k'^8 + \dots \right) \left(\log \frac{4}{k'} - 1 \right) \right. \\
 \left. - \left(1 + \frac{15}{128}k'^4 + \frac{185}{1536}k'^6 + \frac{7465}{65536}k'^8 + \dots \right) \right\} \tag{5}
 \end{aligned}$$

Evidently this expression is rapidly convergent when k' is small, and hence will give an accurate value of M when the circles are near each other. Otherwise formula (1) may be more accurate. An example will be given later to test the correctness of this formula.

NAGAOKA'S FORMULÆ.

Nagaoka³ has given formulæ for the calculation of the mutual inductance of coaxial circles, without the use of tables of elliptic integrals. These formulæ make use of Jacobi's q -series, which is very rapidly convergent. The first is to be used when the circles are not near each other, the second when they are near each other. Either may be employed for a considerable range of distances between the extremes, although the first is more convenient. The first formula is as follows:

² Wied. Ann. 21, p. 344; 1884.

³ Phil. Mag., 6, p. 19; 1903.

$$\begin{aligned} M &= 16\pi^2 \sqrt{Aa} q^3 (1 + \epsilon) \\ &= 4\pi \sqrt{Aa} \{4\pi q^3 (1 + \epsilon)\} \end{aligned} \quad (6)$$

where A and a are the radii of the two circles. ϵ is a correction term which can be neglected when the circles are quite far apart.

$$\epsilon = 3q^4 - 4q^6 + 9q^8 - 12q^{10} + \dots$$

$$q = \frac{l}{2} + \left(\frac{l}{2}\right)^5 + 15\left(\frac{l}{2}\right)^9 + \dots$$

$$l = \frac{1 - \sqrt{k'}}{1 + \sqrt{k'}} \quad k' = \frac{r_2}{r_1} = \frac{\sqrt{(A-a)^2 + d^2}}{\sqrt{(A+a)^2 + d^2}}$$

d being the distance between the centers of the circles, and k' the complementary modulus occurring in equations (3), (4), and (5).

Nagaoka's second formula is as follows:

$$M = 4\pi \sqrt{Aa} \cdot \frac{1}{2(1-2q_1)^3} \left\{ \log \frac{1}{q_1} [1 + 8q_1(1 - q_1 + 4q_1^2)] - 4 \right\} \quad (7)$$

$$q_1 = \frac{l_1}{2} + 2\left(\frac{l_1}{2}\right)^5 + 15\left(\frac{l_1}{2}\right)^9 + \dots$$

$$l_1 = \frac{1 - \sqrt{k}}{1 + \sqrt{k}} \quad k = \frac{2\sqrt{Aa}}{\sqrt{(A+a)^2 + d^2}}$$

k is the modulus of equation (1), but is employed here to obtain the value of the q -series instead of the values of the elliptic integrals employed in (1). This formula is ordinarily simpler in use than it appears, because some of the terms in the expressions above are usually negligibly small.

Examples will be given later illustrating the use of these formulæ.

MAXWELL'S SERIES FORMULA.

Maxwell⁴ obtained an expression for the mutual inductance between two coaxial circles in the form of a converging series which is often more convenient to use than the elliptical integral formula, and when the circles are nearly of the same radii and relatively near each other the value given is generally sufficiently exact. In the following formula a is the smaller of the two radii, c is their

⁴Electricity and Magnetism, Vol. II, § 705.

difference, $A-a$, d is the distance apart of the circles as before, and $r = \sqrt{c^2 + d^2}$. The mutual inductance is then

$$M = 4\pi a \left\{ \log \frac{8a}{r} \left(1 + \frac{c}{2a} + \frac{c^2 + 3d^2}{16a^2} - \frac{c^3 + 3cd^2}{32a^3} + \dots \right) - \left(2 + \frac{c}{2a} - \frac{3c^2 - d^2}{16a^2} + \frac{c^3 - 6cd^2}{48a^3} - \dots \right) \right\} \quad (8)$$

When the two radii are equal, as is often the case in practice, the equation is considerably simplified, as follows:

$$M = 4\pi a \left\{ \log \frac{8a}{d} \left(1 + \frac{3d^2}{16a^2} \right) - \left(2 + \frac{d^2}{16a^2} \right) \right\} \quad (9)$$

The above formulæ (8) and (9) are sufficiently exact for very many cases, the terms omitted in the series being unimportant when $\frac{c}{a}$ and $\frac{d}{a}$ are small. For example, if $\frac{d}{a}$ is 0.1, the largest term neglected in

(9) is less than two parts in a million. If, however $d=a$, this term will be more than one per cent, and the formula will be quite inexact.

Coffin⁵ has extended Maxwell's formula (9) for two equal circles by computing three additional terms for each part of the expression. This enables the mutual inductance to be computed with considerable exactness up to $d=a$. Formula (1) is exact, as stated above, for all distances, and either it or (6) should be used in preference to (10) when d is large. Coffin's formula is as follows:

$$M = 4\pi a \left\{ \log \frac{8a}{d} \left(1 + \frac{3d^2}{16a^2} - \frac{15d^4}{8 \cdot 128a^4} + \frac{35d^6}{128^2a^6} - \frac{1575d^8}{2 \cdot 128^3a^8} + \dots \right) - \left(2 + \frac{d^2}{16a^2} - \frac{31d^4}{16 \cdot 128a^4} + \frac{247d^6}{6 \cdot 128^2a^6} - \frac{7795d^8}{8 \cdot 128^3a^8} + \dots \right) \right\} \quad (10)$$

We have extended Maxwell's formula (8) for unequal circles by expanding the terms of formula (1) in series and substituting these series in (1). The series (3) and (4) involve k' , where $k'^2 = 1 - k^2$. Hence $1 + k'^2 = 2 - k^2$.

Also

$$r = \sqrt{x^2 + y^2}$$

$$k^2 = \frac{4Aa}{(A+a)^2 + x^2}$$

⁵ Bulletin of Bureau of Standards, 2, p. 113; 1906.

$$k'^2 = \frac{x^2 + y^2}{(2a + y)^2 + x^2}$$

$$\frac{\sqrt{Aa}}{k} = \frac{1}{2} \sqrt{(2a + y)^2 + x^2} = a \sqrt{1 + \frac{y}{a} + \frac{x^2 + y^2}{4a^2}}$$

Equation (1) is therefore equal to

$$M = 4\pi a \sqrt{1 + \frac{y}{a} + \frac{x^2 + y^2}{4a^2}} \left\{ \left(1 + k'^2 \right) F - 2E \right\} \quad (a)$$

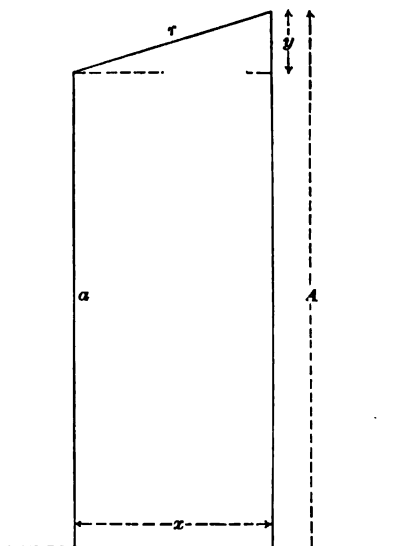


Fig. 2

Substituting the values of F and E from (3) and (4) we get

$$\left\{ \left(1 + k'^2 \right) F - 2E \right\} = \log \frac{4}{k'} \left\{ 1 + \frac{k'^2}{4} + \frac{k'^4}{64} + \dots \right\} - 2 + \frac{k'^2}{4} - \frac{k'^4}{128} + \dots \quad (b)$$

$$= C \log \frac{4}{k'} + D, \text{ say}$$

$$\begin{aligned} \log \frac{4}{k'} &= \log \left(\frac{8a}{r} \sqrt{\left(1 + \frac{y}{2a} \right)^2 + \frac{x^2}{4a^2}} \right) = \log \frac{8a}{r} + \log \sqrt{1 + \frac{y}{2a} + \frac{x^2 + y^2}{4a^2}} \\ &= \log \frac{8a}{r} + \frac{1}{2} \left(\frac{y}{a} + \frac{x^2 + y^2}{4a^2} \right) - \frac{1}{4} \left(\frac{y}{a} + \frac{x^2 + y^2}{4a^2} \right)^2 \\ &\quad + \frac{1}{6} \left(\frac{y}{a} + \frac{x^2 + y^2}{4a^2} \right)^3 - \dots \end{aligned}$$

$$\text{Or, } \log \frac{4}{k'} = \log \frac{8a}{r} + \frac{y}{2a} + \frac{x^2 - y^2}{8a^2} - \frac{6x^2y - 2y^3}{48a^3} - \frac{x^4 - 6x^2y^2 + y^4}{64a^4} + \frac{5x^4y - 10x^2y^3 + y^5}{160a^5} \quad (c)$$

$$= \log \frac{8a}{r} + B, \text{ say}$$

$$k'^2 = \frac{x^2 + y^2}{4a^2} \left(1 + \frac{y}{a} + \frac{x^2 + y^2}{4a^2} \right)^{-1} \\ = \frac{x^2 + y^2}{4a^2} - \frac{x^2y + y^3}{4a^3} - \frac{x^4 - 2x^2y^2 - 3y^4}{16a^4} + \frac{x^4y - y^5}{8a^5} \dots \quad (d')$$

Substituting this value of k'^2 above we have

$$C = 1 + \frac{x^2 + y^2}{16a^2} - \frac{x^2y + y^3}{16a^3} - \frac{15x^4 - 34x^2y^2 - 49y^4}{1024a^4} + \frac{15x^4y - 2x^2y^3 - 17y^5}{512a^5} - \dots \quad (e)$$

$$D = -2 + \frac{x^2 + y^2}{16a^2} - \frac{x^2y + y^3}{16a^3} - \frac{33x^4 - 62x^2y^2 - 95y^4}{2048a^4} + \frac{33x^4y + 2x^2y^3 - 31y^5}{1024a^5} \dots \quad (f)$$

$$\left(1 + \frac{y}{a} + \frac{x^2 + y^2}{4a^2} \right)^{\frac{1}{2}} = 1 + \frac{y}{2a} + \frac{x^2}{8a^2} - \frac{x^2y}{16a^3} + \frac{4x^2y^2 - x^4}{128a^4} + \frac{6x^4y - 8x^2y^3}{512a^5} - \dots = A, \text{ say} \quad (g)$$

Equation (a) is now

$$M = 4\pi a \cdot A \left[C \log \frac{8a}{r} + BC + D \right] \quad (h)$$

in which A , B , C , and D are the series (g , c , e , f) given above. Substituting in (h) and omitting terms of higher than the fifth degree we get

$$M = 4\pi a \left\{ \log \frac{8a}{r} \left(1 + \frac{y}{2a} + \frac{3x^2 + y^2}{16a^2} - \frac{3x^2y + y^3}{32a^3} - \frac{15x^4 - 42x^2y^2 - 17y^4}{1024a^4} + \frac{45x^4y - 30x^2y^3 - 19y^5}{2048a^5} + \dots \right) - \left(2 + \frac{y}{2a} + \frac{x^2 - 3y^2}{16a^2} - \frac{6x^2y - y^3}{48a^3} - \frac{93x^4 - 534x^2y^2 - 19y^4}{6144a^4} + \frac{1845x^4y - 3030x^2y^3 - 379y^5}{61440a^5} + \dots \right) \right\} \quad (11)$$

When $y=0$, this gives the first part of series (10). When $x=0$, the case of two circles in the same plane, with radii a and $a+y$, we have

$$M=4\pi a \left\{ \log \frac{8a}{y} \cdot \left(1 + \frac{y}{2a} + \frac{y^2}{16a^2} - \frac{y^3}{32a^3} + \frac{17y^4}{1024a^4} - \frac{19y^5}{2048a^5} - \dots \right) \right. \\ \left. - \left(2 + \frac{y}{2a} - \frac{3y^2}{16a^2} + \frac{y^3}{48a^3} + \frac{19y^4}{6144a^4} - \frac{379y^5}{61440a^5} - \dots \right) \right\} \quad (12)$$

In the above formulæ x and y are interchanged from Maxwell's notation and correspond to d and c of (8). That is, x is the distance between the centers of the circles, and y is the excess of one radius over a , the radius of the other; y may be $+$ or $-$.

These formulæ give the mutual inductance with great precision when the coils are not too far apart. The degree of convergence, of course, indicates in any case about what the limit of accuracy is.

We have derived equation (11) also by the method given by Maxwell,⁶ to check the coefficients.

MUTUAL INDUCTANCE OF TWO COAXIAL COILS.

ROWLAND'S FORMULA.

Let there be two coaxial coils of mean radii A and a , axial breadth of coils b_1 and b_2 , radial depth c_1 and c_2 , and distance apart of their mean planes d . Suppose them uniformly wound with n_1 and n_2 turns of wire. The mutual inductance M_0 of the two central turns, O_1 and O_2 (Fig. 3), will be given by formula (1) or (5), and the mutual inductance M of the two coils of n_1 and n_2 turns will then be, to a *first approximation*,

$$M=n_1n_2M_0$$

A second approximation was obtained by Rowland by means of Taylor's theorem, following Maxwell, § 700. The mutual inductance of the two central turns O_1 and O_2 , being M_0 , the mutual inductance of O_1 on any turn at P of coordinates x, y in coil B is given by Taylor's theorem, as follows:

⁶ Electricity and Magnetism, Vol. II, § 705.

$$M = M_0 + x \frac{dM_0}{dx} + y \frac{dM_0}{dy} + \frac{x^2}{2} \frac{d^2 M_0}{dx^2} + \frac{y^2}{2} \frac{d^2 M_0}{dy^2} + xy \frac{d^2 M_0}{dxdy} + \dots (i)$$

If we integrate this expression over the area of the coil B to find the equivalent value M' for the whole area, where

$$M'xy = \int_{-\frac{x}{2}}^{+\frac{x}{2}} \int_{-\frac{y}{2}}^{+\frac{y}{2}} M dx dy$$

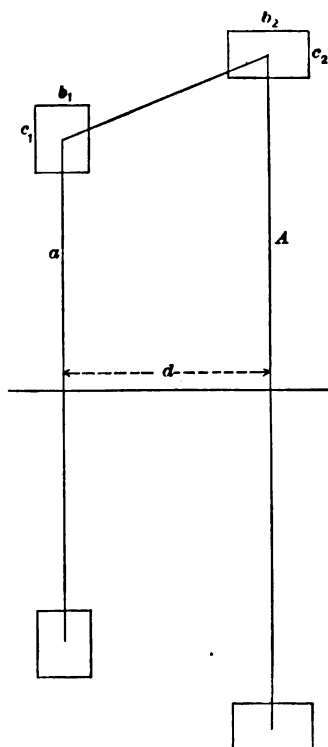


Fig. 8

we have (since the second and third terms become zero)

$$M' = M_0 + \frac{x^2}{24} \frac{d^2 M_0}{dx^2} + \frac{y^2}{24} \frac{d^2 M_0}{dy^2}$$

neglecting terms in $\frac{d^3 M_0}{dx^3}$, $\frac{d^3 M_0}{dx^2 dy}$ and $\frac{d^3 M_0}{dy^3}$ and higher orders.

Substituting b_2 and c_2 for x and y , and da for dy , we have

$$M' = M_0 + \frac{b_2^2}{24} \frac{d^2 M_0}{dx^2} + \frac{c_2^2}{24} \frac{d^2 M_0}{da^2} \quad (k)$$

$n_2 M'$ is the mutual inductance of a single turn at O_1 and the coil B. Repeating the process, integrating over A, we get M , $n_1 n_2 M$ being the mutual inductance of one coil on the other. Thus,

$$M = M_0 + \frac{1}{24} \left\{ (b_1^2 + b_2^2) \frac{d^2 M_0}{dx^2} + c_1^2 \frac{d^2 M_0}{da^2} + c_2^2 \frac{d^2 M_0}{da^2} \right\}$$

If the two coils are of equal radii but unequal section,

$$M = M_0 + \frac{1}{24} \left\{ (b_1^2 + b_2^2) \frac{d^2 M_0}{dx^2} + (c_1^2 + c_2^2) \frac{d^2 M_0}{da^2} \right\} \quad (13)$$

If the two coils are of equal radii and equal section, this becomes

$$M = M_0 + \frac{1}{12} \left\{ b^2 \frac{d^2 M}{dx^2} + c^2 \frac{d^2 M}{da^2} \right\} \quad (14)$$

The value of M_0 should be calculated by formula (1). The correction terms will be calculated by means of the following:

$$\left. \begin{aligned} \frac{d^2 M_0}{da^2} &= \pi \frac{k}{a} \left\{ (2 - k^2) F - \left(2 - k^2 \frac{1 - 2k^2}{1 - k^2} \right) E \right\} \\ \frac{d^2 M_0}{dx^2} &= \pi \frac{k^3}{a} \left\{ F - \frac{1 - 2k^2}{1 - k^2} E \right\} \end{aligned} \right\} \quad (15)$$

The equation (14) is equivalent to Rowland's equation, where 2ξ and 2η are the breadth and depth of the section of the coil, instead of b and c , except that there is an error in the formula as printed in Rowland's⁷ paper, ξ and η being interchanged. The equations (15) are equivalent to those given by Rowland, being somewhat simpler.⁸

Formula (12) gives a very exact value for the mutual inductance of two coils, provided the cross sections are relatively small and the distance apart d is not too small. But when b or c is large or d is small the fourth differential coefficients which have been neglected become appreciable and the expression may not be sufficiently exact.

⁷ Collected Papers, p. 162.

⁸ Gray, Absolute Measurements, Vol. II, Part II, p. 322.

RAYLEIGH'S FORMULA.

Maxwell⁹ gives a formula, suggested by Rayleigh, for the mutual inductance of two coils, which has a very different form from Rowland's, but is nearly equivalent to it when the coils are not near each other. It has been used by Glazebrook and Rayleigh, and may also be employed in calculating the attraction between two coils.¹⁰ It is sometimes called the formula of quadratures. It is derived as follows:

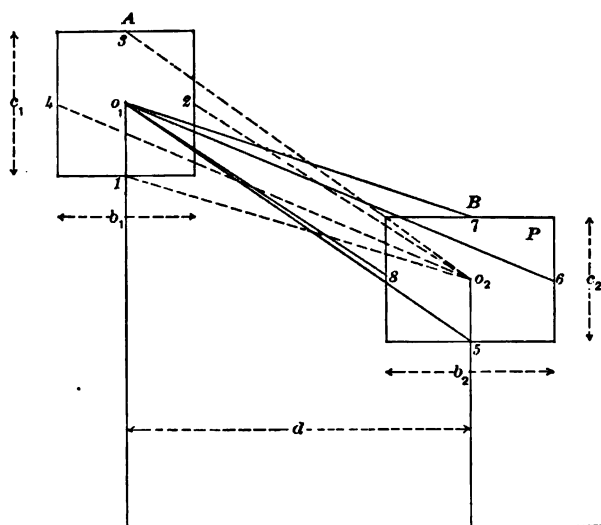


Fig. 4

Referring to Fig. 4, let M_1 , M_3 be the mutual inductances of the central wire O_2 and the wires at points 1 and 3, respectively, of coil A. For these points $x=0$, and y is $-\frac{c_1}{2}$ and $+\frac{c_1}{2}$, respectively. Substituting these values of x and y in equation (1) we have,

$$M_1 = M_0 - \frac{c_1}{2} \frac{dM_0}{da} + \frac{c_1^2}{8} \frac{d^2M_0}{da^2} - \frac{c_1^3}{48} \frac{d^3M_0}{da^3} + \frac{c_1^4}{384} \frac{d^4M_0}{da^4} - \dots$$

$$M_3 = M_0 + \frac{c_1}{2} \frac{dM_0}{da} + \frac{c_1^2}{8} \frac{d^2M_0}{da^2} + \frac{c_1^3}{48} \frac{d^3M_0}{da^3} + \frac{c_1^4}{384} \frac{d^4M_0}{da^4} + \dots$$

⁹ Electricity and Magnetism, Vol. II, Appendix II, Chapter XIV.

¹⁰ Gray, Absolute Measurements, Vol. II, Part II, p. 403.

$$\left. \begin{aligned} \text{whence, } M_1 + M_3 - 2M_0 &= \frac{c_1^2}{4} \frac{d^2 M_0}{da^2} + \frac{c_1^4}{192} \frac{d^4 M_0}{da^4} + \dots \\ \text{Similarly, } M_2 + M_4 - 2M_0 &= \frac{b_1^2}{4} \frac{d^2 M_0}{dx^2} + \frac{b_1^4}{192} \frac{d^4 M_0}{dx^4} + \dots \end{aligned} \right\} (l)$$

In a similar manner, if M_5, M_6, M_7, M_8 are the coefficients of inductance of the single turn O_1 , on single turns at points 5, 6, 7, 8 of coil B, we get

$$\left. \begin{aligned} M_5 + M_7 - 2M_0 &= \frac{c_2^2}{4} \frac{d^2 M_0}{dA^2} + \frac{c_2^4}{192} \frac{d^4 M_0}{dA^4} + \dots \\ M_6 + M_8 - 2M_0 &= \frac{b_2^2}{4} \frac{d^2 M_0}{dx^2} + \frac{b_2^4}{192} \frac{d^4 M_0}{dx^4} + \dots \end{aligned} \right\} (m)$$

Adding equations (l) and (m) we have

$$\begin{aligned} & \frac{1}{6} (M_1 + M_3 + M_5 + M_7 + M_2 + M_4 + M_6 + M_8 - 2M_0) \\ &= M_0 + \frac{1}{24} \left\{ (b_1^2 + b_2^2) \frac{d^2 M_0}{dx^2} + c_1^2 \frac{d^2 M_0}{da^2} + c_2^2 \frac{d^2 M_0}{dA^2} \right\} \\ &+ \frac{1}{192} \left\{ (b_1^4 + b_2^4) \frac{d^4 M_0}{dx^4} + c_1^4 \frac{d^4 M_0}{da^4} + c_2^4 \frac{d^4 M_0}{dA^4} \right\} + \dots \end{aligned} \quad (n)$$

The mutual inductance of two coils of unequal radii and unequal section is, neglecting sixth and higher differentials,¹¹

$$\begin{aligned} M &= M_0 + \frac{1}{24} \left\{ (b_1^2 + b_2^2) \frac{d^2 M_0}{dx^2} + c_1^2 \frac{d^2 M_0}{da^2} + c_2^2 \frac{d^2 M_0}{dA^2} \right\} \\ &+ \frac{1}{1920} \left\{ (b_1^4 + b_2^4) \frac{d^4 M_0}{dx^4} + c_1^4 \frac{d^4 M_0}{da^4} + c_2^4 \frac{d^4 M_0}{dA^4} \right\} \\ &+ \frac{1}{576} \left\{ b_1^2 b_2^2 \frac{d^4 M_0}{dx^4} + c_1^2 c_2^2 \frac{d^4 M_0}{da^2 dA^2} \right\} \\ &+ \frac{1}{576} \left\{ (b_1^2 + b_2^2) \left(c_1^2 \frac{d^4 M_0}{dx^2 da^2} + c_2^2 \frac{d^4 M_0}{dx^2 dA^2} \right) \right\} \end{aligned} \quad (16)$$

¹¹ Rosa, this Bulletin, p. 337.

For two equal coils this is

$$M = M_0 + \frac{1}{12} \left(b^2 \frac{d^2 M_0}{dx^2} + c^2 \frac{d^2 M_0}{da^2} \right) + \frac{1}{360} \left(b^4 \frac{d^4 M_0}{dx^4} + c^4 \frac{d^4 M_0}{da^4} \right) + \frac{b^2 c^2}{144} \frac{d^4 M_0}{dx^2 da^2} \quad (17)$$

Equation (n) agrees with (16) *only when the fourth differentials are negligible*. In that case we can write

$$M = \frac{1}{6} (M_1 + M_2 + M_3 + M_4 + M_5 + M_6 + M_7 + M_8 - 2M_0) \quad (18)$$

For two coils of equal radii and equal section this becomes

$$M = \frac{1}{3} (M_1 + M_2 + M_3 + M_4 - M_0) \quad (19)$$

Equation (18) is Rayleigh's formula, or the formula of quadratures. Instead of computing the correction to M_0 by means of the differential coefficients (13), eight additional values are computed, corresponding to the mutual inductances of the single turns at the eight points indicated in Fig. 4, each with reference to the central turn of the other coil. These M 's may be computed by formulæ (8) and (9) or (10) and (11), and the values of the constants for the case of two coils of equal radii are given in the following table, the radius being a in every case.

	Axial distance.	Radial distance.	r
Using (8)	d	$-\frac{c_1}{2}$	$\sqrt{d^2 + \frac{c_1^2}{4}}$
"	d	$+\frac{c_1}{2}$	$\sqrt{d^2 + \frac{c_1^2}{4}}$
"	d	$-\frac{c_2}{2}$	$\sqrt{d^2 + \frac{c_2^2}{4}}$
"	d	$+\frac{c_2}{2}$	$\sqrt{d^2 + \frac{c_2^2}{4}}$
Using (9)	$d - b_1/2$	0	
"	$d + b_1/2$	0	
"	$d + b_2/2$	0	
"	$d - b_2/2$	0	

MAGNITUDE OF THE ERRORS IN ROWLAND'S AND RAYLEIGH'S FORMULÆ.

The error in equation (18) is the difference between the values of M as given by (n) and (16). Calling this correction ϵ_1 , and taking the difference for simplicity for the case of two *equal coils*,

$$\epsilon_1 = \frac{1}{960} \left\{ b^4 \frac{d^4 M_0}{dx^4} + c^4 \frac{d^4 M_0}{da^4} \right\} + \frac{b^3 c^3}{144} \frac{d^4 M_0}{dx^3 da^3} \quad (o)$$

The values of the differential coefficients of (17) are as follows:¹³

$$\begin{aligned} \frac{d^4 M_0}{dx^4} &= \frac{d^4 M_0}{da^4} = 4\pi a \times \frac{6}{d^4} \\ \frac{d^4 M}{dx^3 da^3} &= -4\pi a \times \frac{6}{d^4} \end{aligned}$$

Substituting these values in (o) we have

$$\epsilon_1 = 4\pi a \left\{ \frac{3b^4 + 3c^4 - 20b^3 c^3}{480d^4} \right\} \quad (20)$$

For a square coil the correction is a negative quantity, showing that M by equation (19) is too large, and the error is proportional to the fourth power of $\frac{1}{d}$, the reciprocal of the distance between the mean planes of the coils. For a rectangular coil in which b is greater than c the correction is negative so long as b is not more than 2.5 times c . When b is still larger with respect to c the correction becomes plus, the value of M by (19) being too small.

Thus, for a coil of cross section 4 sq. cm, we get the following values of the numerator of (20) as we vary the shape of cross section, keeping $bc=4$.

Dimensions of coil.	Error proportional to—
$b=2 \quad c=2$	— 224
$b=2.5 \quad c=1.6$	— 183
$b=3 \quad c=1.33$	— 67.5
$b=4 \quad c=1$	+ 451
$b=8 \quad c=0.5$	+ 11,988

Thus we see that the value of M as given by the formula of quadratures may be too large or too small according to the shape of the

¹³Rosa, this Bulletin, p. 346.

section, and that the error is proportional to the fourth power of the dimensions of the section divided by the distance between the mean planes of the coils. When the section is small and d large the error will become negligible.

The error by Rowland's formula is found by taking the difference between (14) and (17). Thus the error ϵ_2 is

$$\epsilon_2 = 4\pi a \frac{6}{d^4} \left\{ \frac{b^4 + c^4}{360} - \frac{b^2 c^2}{144} \right\} = 4\pi a \frac{8b^4 + 8c^4 - 20b^2 c^2}{480a^4} \quad (21)$$

This is negative for a square coil, but smaller than ϵ_1 . For a coil of section such that $b = c\sqrt{2}$, this error is zero, and for sections such that that $\frac{b}{c} > \sqrt{2}$, the error is positive. Thus, for a coil of cross section 4 sq. cm, we get the following values of the numerator of (21) which is proportional to the error by Rowland's formula.

Dimensions of coil.	Error proportional to—
$b = 2 \quad c = 2$	— 64
$b = 2.5 \quad c = 1.6$	+ 45
$b = 3 \quad c = 1.33$	+ 353
$b = 4 \quad c = 1$	+ 1,736
$b = 8 \quad c = 0.5$	+ 32,448

Thus the error is smaller by Rowland's formula for coils having square or nearly square section, but larger for coils having rectangular sections not nearly square.

These conclusions are verified by numerical calculations with the formulæ of Rowland and Rayleigh later in this paper.

LYLE'S FORMULA.

Professor Lyle¹³ has recently proposed a very convenient method for calculating the mutual inductance of coaxial coils, which gives very accurate results for coils at some distance from each other. Lyle begins his demonstration with Maxwell's¹⁴ expression for the magnetic potential of a coil at any point in its axis, namely,

$$V = 2\pi n C \left\{ 1 - \frac{x}{\rho} + \frac{c^2 x}{24\rho^3} + \frac{b^2 - c^2}{8\rho^5} \cdot a^2 x \right\} \quad (p)$$

¹³ Phil. Mag., 8, p. 310; 1902.

¹⁴ Electricity and Magnetism, Vol. II, § 700.

assuming the dimensions of the cross section small in comparison with the radius of the coil, and the winding uniform.

In the above equation,

a = mean radius of the coil,

b = axial breadth,

c = radial depth,

x = distance on axis from center of coil,

$\rho = \sqrt{a^2 + x^2}$

C is the current, and n is the number of turns in the coil. This

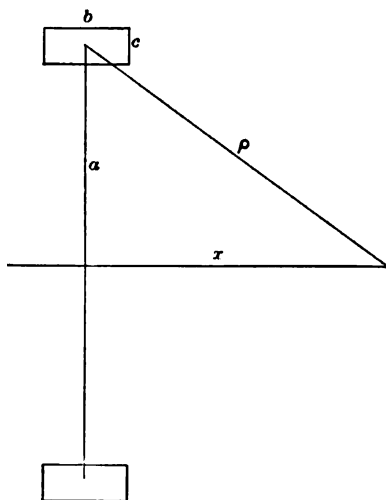


Fig. 5

notation differs from Lyle's only in using, as elsewhere in this article, b and c for the breadth and depth of coil, instead of ξ and η .

Expanding the above equation in ascending powers of $\frac{x}{a}$, Lyle obtains,

$$V = 2\pi n C \left[1 - \frac{x}{a} \left(1 - \frac{3b^2 - 2c^2}{24a^2} \right) + \frac{x^3}{2a^3} \left(1 - \frac{3(5b^2 - 4c^2)}{24a^2} \right) - \frac{3x^5}{2.4a^5} \left(1 - \frac{5(7b^2 - 6c^2)}{24a^2} \right) + \frac{3.5x^7}{2.4.6a^7} \left(1 - \frac{7(9b^2 - 8c^2)}{24a^2} \right) - \dots \right] \quad (9)$$

The potential V' of a circular filament of radius r and carrying current nC at a point P distant x from the center of the circle is

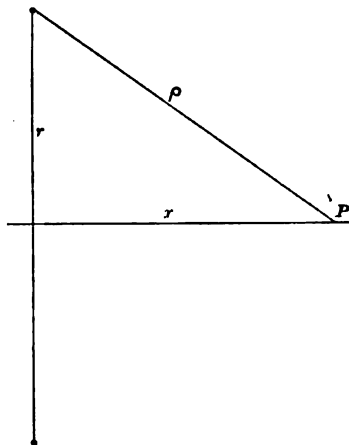


Fig. 6

$$V' = 2\pi nC \left\{ 1 - \frac{x}{\sqrt{x^2 + r^2}} \right\} \quad (r)$$

which expanded in ascending powers of $\frac{x}{r}$ gives

$$V' = 2\pi nC \left\{ 1 - \frac{x}{r} + \frac{x^3}{2r^3} - \frac{1 \cdot 3x^5}{2 \cdot 4r^5} + \frac{1 \cdot 3 \cdot 5x^7}{2 \cdot 4 \cdot 6r^7} - \dots \right\} \quad (s)$$

The two potentials V in (q) and V' in (s) will be identical, provided

$$\left. \begin{aligned} \frac{1}{r} &= \frac{1}{a} \left\{ 1 - \frac{3b^2 - 2c^2}{24a^2} \right\} \\ \frac{1}{r^3} &= \frac{1}{a^3} \left\{ 1 - \frac{3(5b^2 - 4c^2)}{24a^2} \right\} \\ \frac{1}{r^5} &= \frac{1}{a^5} \left\{ 1 - \frac{5(7b^2 - 6c^2)}{24a^2} \right\} \end{aligned} \right\} \quad (t)$$

If the section is square, and hence $b=c$, these equations become

$$\left. \begin{aligned} \frac{1}{r} &= \frac{1}{a} \left(1 - \frac{b^2}{24a^2} \right) \\ \frac{1}{r^3} &= \frac{1}{a^3} \left(1 - \frac{3b^2}{24a^2} \right) \\ \frac{1}{r^5} &= \frac{1}{a^5} \left(1 - \frac{5b^2}{24a^2} \right) \end{aligned} \right\} \quad (u)$$

Cubing the first of these equations we have

$$\frac{1}{r^3} = \frac{1}{a^3} \left(1 - \frac{3b^2}{24a^2} + \frac{3b^4}{24^3 a^4} - \frac{b^6}{24^3 a^6} \right)$$

Hence, if we neglect the terms in $\frac{b^4}{a^4}$ and higher powers of $\left(\frac{b}{a}\right)$ we see that all the equations of condition (t) are satisfied by making $b=c$. The same result follows if the expansions are in ascending powers of $\frac{a}{x}$.

The first of equations (u) gives

$$r = a \left(1 + \frac{b^2}{24a^2} \right) \quad (22)$$

again neglecting fourth and higher powers of $\frac{b}{a}$. Hence, we see that

if a coil of radius a , cross section b^2 , wound with n turns of wire and carrying a current C , be replaced by a single filament lying in the mean plane of the coil, of radius r and carrying a current nC , the magnetic potential will be identical at all points on the axis, and hence, by Legendre's theorem, identical at all points of space without the coil. Thus, this filament is seen to be equivalent to the coil and can replace it so far as its *external field is concerned*. If there were n turns in the filament O, through which current C flows, the flux due to O through coil B would be the same as that due to current C through the n turns of A.

But since the mutual inductance is the same whichever coil is the primary, the flux through O due to current C in B must be the same as it is through the n turns of A, and therefore the filament O can replace the coil A, not only so far as its own external field is concerned, but also so far as the effect of external fields on it is concerned.

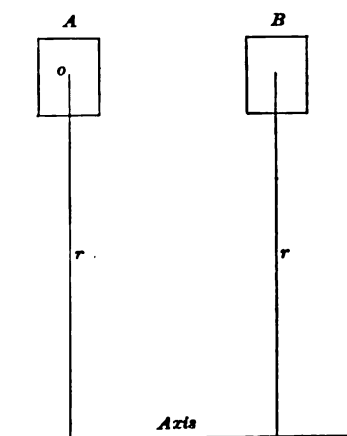


Fig. 7

In the proof of equation (p), however, differential coefficients of the fourth and higher orders have been neglected, and hence when we apply Legendre's theorem and pass to space off the axis we must keep away from the immediate region of the coil itself, where the fourth and higher differentials of the magnetic potential can not be neglected. Obviously a single filament can not replace a coil of rectangular section *just outside the coil*, and we shall see later that it does so to a high order of approximation only at some distance from it.

Lyle then goes on to show that a coil of rectangular section not square can be replaced by two filaments, the distance apart of the filaments being called the *equivalent breadth* or the *equivalent depth* of the coil.

$$\left. \begin{aligned} \beta^2 &= \frac{b^2 - c^2}{12}, \quad 2\beta \text{ is the equivalent breadth of A} \\ \delta^2 &= \frac{c^2 - b^2}{12}, \quad 2\delta \text{ is the equivalent depth of B} \end{aligned} \right\} \quad (23)$$

The equivalent radius of A is given by the same expression which holds for a square coil, viz:

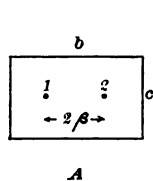
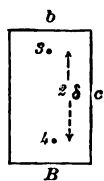


Fig. 8



$$r = a \left(1 + \frac{c^2}{24a^2} \right)$$

In the coil B the equivalent filaments have radii $r + \delta$ and $r - \delta$, respectively, where

$$r = a \left(1 + \frac{b^2}{24a^2} \right)$$

The mutual inductance of two coils may now be readily calculated. If each has a square section, it is necessary only to calculate the mutual inductance of the two equivalent filaments. For coils of rectangular sections, as A, B, the mutual inductance will be the sum of the mutual inductances of the two filaments of A on the two filaments of B, counting $n/2$ turns in each. Or, it is $n_1 n_2$ times the mean of the four inductances $M_{13}, M_{14}, M_{23}, M_{24}$, where M_{13} is the mutual inductance of filament 1 on filament 3, etc.

Similarly the attractions of coils when carrying currents may be calculated.

In a uniform magnetic field the equivalent radius of a coil is easily found as follows. The mutual inductance of two coils is proportional to the whole number of lines of force due to A linked with the various turns of coil B. For a uniform field this is proportional to the sum of the areas of the various turns of the coil. We can therefore find the equivalent radius r_0 for a coil of rectangular section by integration, r_1 and r_2 being the inner and outer radii of the coil. Thus—

$$\pi r_0^2 = \frac{\pi}{r_2 - r_1} \int_{r_1}^{r_2} r^2 dr = \frac{\pi}{r_2 - r_1} \left[\frac{r_2^3 - r_1^3}{3} \right]$$

$$\therefore r_0^2 = \frac{1}{3} (r_1^2 + r_1 r_2 + r_2^2) = \left(\frac{r_1 + r_2}{2} \right)^2 \left[1 + \frac{1}{3} \left(\frac{r_2 - r_1}{r_2 + r_1} \right)^2 \right]$$

If a is the mean radius of the coil and c is the radial depth

$$a = \frac{r_1 + r_2}{2}$$

$$c = r_2 - r_1$$

$$\therefore r_0^2 = a^2 \left(1 + \frac{c^2}{12a^2} \right)$$

$$\text{or } r_0 = a \left(1 + \frac{c^2}{24a^2} \right)$$

neglecting terms in the fourth and higher powers of $\left(\frac{c}{a}\right)$. This value of the equivalent radius which applies to any coil of rectangular section in a *uniform field* is exactly the value found above for a coil of *square section* in a non-uniform field, where fourth and higher differentials are negligible.

Lyle states that in his method of obtaining the mutual inductance no quantities are neglected of order lower than the fourth in $\left(\frac{b}{a}\right)$. It is to be observed, however, that that statement only applies when the coils are a considerable distance apart, as the term neglected depend-

ing on the fourth differentials is proportional to $\left(\frac{b}{d}\right)^4$, and this may be much larger than $\left(\frac{b}{a}\right)^4$.

Lyle's method is of special value in computing mutual inductances because it applies to coils of unequal as well as of equal radii. Examples and tests of the method will be given later.

We will now deduce an expression for ΔM based on Lyle's value of the equivalent radius (22). Thus, putting a_1 for the equivalent radius where a is the mean radius, we have as before

$$a_1 = a \left(1 + \frac{b^2}{24a^2} \right)$$

We may use this value of the radius in any formula for the mutual inductance of two coaxial circles, as 1, 2, 5, or 9. Substituting in (9) we have for M and M_0 ,

$$M = 4\pi a_1 \left\{ \log \frac{8a_1}{d} \left(1 + \frac{3}{16} \frac{d^2}{a_1^2} - \dots \right) - \left(2 + \frac{d^2}{16a_1^2} - \dots \right) \right\}$$

$$M_0 = 4\pi a \left\{ \log \frac{8a}{d} \left(1 + \frac{3}{16} \frac{d^2}{a^2} - \dots \right) - \left(2 + \frac{d^2}{16a^2} - \dots \right) \right\}$$

The correction ΔM is found by taking the difference between M and M_0 . Thus,

$$\begin{aligned} \Delta M &= 4\pi(a_1 - a) \left\{ \log \frac{8a}{d} \left(1 + \frac{3}{16} \frac{d^2}{a^2} \right) - \left(2 + \frac{d^2}{16a^2} \right) \right\} \\ &\quad + 4\pi a \left\{ \log \frac{a_1}{a} \left(1 + \frac{3}{16} \frac{d^2}{a^2} \right) - \log \frac{8a}{d} \left(\frac{3d^2 a_1^2 - a^2}{16 a_1^2} \right) + \frac{d^2}{16} \frac{a_1^2 - a^2}{a_1^2} \right\} \\ &= 4\pi a \left\{ \frac{b^2}{24a^2} \left[\left(\log \frac{8a}{d} - 2 \right) + \frac{3}{16a^2} \log \frac{8a}{d} - \frac{d^2}{16a^2} \right] \right. \\ &\quad \left. + \frac{b^2}{24a^2} + \frac{b^2 d^2}{128a^4} - \frac{3b^2 d^2}{192a^4} \log \frac{8a}{d} + \frac{b^2 d^2}{192a^4} \right\} \quad (v) \end{aligned}$$

Since $\frac{a_1 - a}{a} = \frac{b^2}{24a^2}$ by assumption,

$$a_1^2 - a^2 = \frac{b^2}{12}, \text{ approximately,}$$

and $\log \frac{a_1}{a} = \frac{b^2}{24a^2}$, approximately.

Combining terms in (v)

$$\Delta M = 4\pi a \left\{ \frac{b^2}{24a^2} \left(\log \frac{8a}{d} - 1 \right) - \frac{b^2 d^2}{128a^4} \left(\log \frac{8a}{d} - \frac{4}{3} \right) \right\}$$

$$\text{or} \quad \frac{\Delta M}{\pi} = \frac{b^2}{6a} \left\{ \log \frac{8a}{d} - 1 - \frac{3d^2}{16a^2} \left(\log \frac{8a}{d} - \frac{4}{3} \right) \right\} \quad (24)$$

Comparing this equation with (27) we see that it differs only in the absence of the two terms $-\frac{a^2 b^2}{5a^2}$ and $+\frac{17b^2}{240a^2}$. These terms depend on the fourth differentials, which, as stated above, were ignored in deriving equation (p). Thus we see that, if in deriving equation (27) we had ignored fourth differentials, we should have come to the same result that Lyle has, although the process is very different and the form of the result is very different. Since Rowland's formula depends on second differentials only, we should expect it to agree closely with Lyle's, and we shall see presently that for coils of square section it does.

In a similar manner we may obtain an expression for ΔM for two coils of unequal radii. Substituting a_1 , the equivalent radius in (8), and putting y and y_1 for c , we have:

$$\begin{aligned} M &= 4\pi a_1 \left\{ \log \frac{8a_1}{r_1} \left(1 + \frac{y_1}{2a_1} + \frac{y_1^2}{16a_1^2} + \frac{3d^2}{32a_1^3} - \frac{y_1^3}{32a_1^3} + \frac{3y_1 d^2}{32a_1^3} - \dots \right) \right. \\ &\quad \left. - \left(2 + \frac{y_1}{2a} - \frac{3y_1^2 - d^2}{16a_1^2} + \frac{y_1^3 - 6y_1 d^2}{48a_1^3} - \dots \right) \right\} \\ M_0 &= 4\pi a \left\{ \log \frac{8a}{r} \left(1 + \frac{y}{2a} + \frac{y^2}{16a^2} - \frac{y^3}{32a^3} + \frac{3y d^2}{32a^3} - \dots \right) \right. \\ &\quad \left. - \left(2 + \frac{y}{2a} - \frac{3y^2 - d^2}{16a^2} + \frac{y^3 - 6y d^2}{48a^3} \right) \right\} \end{aligned}$$

Putting as before the difference between these two expressions equal to ΔM , we have:

$$\frac{\Delta M}{\pi} = \frac{b^2}{6a} \left\{ \left(\log \frac{8a}{r} + \frac{ay^2}{Ar^2} \right) \left(1 + \frac{y}{2a} + \frac{y^2}{16a^2} - \frac{y^3}{32a^3} + \frac{3y d^2}{32a^3} \right) \right.$$

(Equation continued next page.)

$$\begin{aligned}
& -1 + \frac{2y^2 + d^2}{8a^2} - \frac{5y^3 - 3yd^2}{96a^3} - \frac{(A+a)y}{2Aa} \left(\log \frac{8a}{r} - 1 \right) \\
& - \frac{(A+a)y^2}{8Aa^2} \left(\log \frac{8a}{r} + 3 \right) - \frac{3d^2}{8a^2} \left(\log \frac{8a}{r} - \frac{1}{3} \right) \\
& + \frac{3(A+a)y^2}{32Aa^2} \left(\log \frac{8a}{r} + \frac{2}{3} \right) + \frac{3(3A+a)d^2y}{32Aa^2} \left(\log \frac{8a}{r} - \frac{4}{3} \right) \} \quad (24a)
\end{aligned}$$

This applies to coils of equal square sections, of radii a and A , distance between centers being d ; $y = A - a$.

This formula is easier to use than it might appear to be. There is only one natural logarithm to get, and when one is calculating ΔM directly it is not necessary to work to so high precision as when calculating M . If, however, one wishes only M and not ΔM it would be better to calculate it directly by Lyle's method.

STEFAN'S FORMULA.

Stefan's¹⁵ formula for the mutual inductance of two equal coaxial coils is as follows:

$$\begin{aligned}
M = 4\pi an^2 \left\{ \log \frac{8a}{d} - 2 + \frac{b^2 - c^2}{12d^2} + \frac{2b^4 + 2c^4 - 5b^2c^2}{120d^4} + \frac{3b^6 - 7b^4c^2 + 7b^2c^4 - 3c^6}{504d^6} \right. \\
\left. + \left(\log \frac{8a}{d} - 2 \right) \left(\frac{3b^2 + c^2 + 18d^2}{96a^4} - \frac{15d^4}{1024a^4} \right) + \frac{7b^2 + 23c^2 + 60d^2}{192a^2} - \frac{29d^4}{2048a^4} \right\} \quad (25)
\end{aligned}$$

This formula¹⁶ may be written $M = M_0 + \Delta M$ where M_0 is the mutual inductance of the central circles of the two coils and ΔM is the correction for the section of the coil, but the value of ΔM in formula (25) is incorrect. The corrected expression for ΔM is as follows:¹⁶

$$\begin{aligned}
\frac{\Delta M}{\pi n^2} = 4a \left\{ \frac{3b^2 + c^2}{96a^2} \cdot \log \frac{8a}{d} - \frac{11b^2 - 3c^2}{192a^2} + \frac{b^2 - c^2}{12d^2} + \frac{2b^4 + 2c^4 - 5b^2c^2}{120d^4} \right. \\
+ \frac{6b^4 + 6c^4 + 5b^2c^2}{5760a^2d^2} + \frac{3b^6 - 3c^6 + 14b^2c^4 - 14b^4c^2}{504d^6} + \frac{7c^2d^2}{1024a^4} \left(\log \frac{8a}{d} - \frac{163}{84} \right) \\
\left. - \frac{15b^2d^2}{1024a^4} \left(\log \frac{8a}{d} - \frac{97}{60} \right) \right\} \quad (26)
\end{aligned}$$

¹⁵ Wied. Annalen, 22, p. 107; 1884. ¹⁶ Rosa, this Bulletin, p. 348, (38) and (39).

For a square section, when $b=c$, this becomes

$$\frac{\Delta M}{\pi n^2} = \frac{b^2}{6a} \left\{ \log \frac{8a}{d} - 1 - \frac{a^2 b^2}{5d^4} - \frac{3d^2}{16a^2} \left(\log \frac{8a}{d} - \frac{4}{3} \right) + \frac{17b^2}{240d^2} \right\} \quad (27)$$

The last two terms of equation (27) are relatively small, so that we may write, *approximately*:

$$\frac{\Delta M}{\pi n^2} = \frac{b^2}{6a} \left\{ \log \frac{8a}{d} - 1 - \frac{a^2 b^2}{5d^4} \right\} \quad (28)$$

These expressions for ΔM are very exact where the coils are near together or where they are separated for a considerable distance, but become less exact as d is greater. They are therefore most reliable where formulæ (14), (19), and (22) are least reliable. As formula (28) is exact enough for most purposes, it affords a very easy method of getting the correction for equal coils of square section.

We give later examples to illustrate and test the accuracy of the above formulæ.

WEINSTEIN'S FORMULA.

Weinstein¹⁷ gives a formula for the mutual inductance of equal coaxial coils, as follows:

$$M = C \left\{ (K - E) \left[\epsilon + \frac{\cos^2 \lambda}{12h^2} \left(a_1 - a_2 - a_3 + (2a_2 - 3a_3) \cos^2 \lambda + 8a_3 \cos^4 \lambda \right) \right] \right. \\ \left. - E \left[1 - \frac{\sin^2 \lambda}{12h^2} \left(a_1 + \frac{a_2}{2} + 2a_3 + (2a_2 + 3a_3) \cos^2 \lambda + 8a_3 \cos^4 \lambda \right) \right] \right\} \quad (29)$$

where $C = 2\pi d n_1 n_2 \sin \lambda$, d is the mean diameter of the coil, and K and E are elliptic integrals. The above expression also is of the form $M = M_0 + \Delta M$, and can be used better in the second form, where M_0 can be calculated by any one of several reliable formulæ and ΔM can be computed separately. The expression for ΔM derived from (29) is as follows:¹⁸

$$\frac{\Delta M}{\pi n_1 n_2} = 4a \sin \gamma \left\{ (F - E) \left(A + \frac{c^2}{24a^2} \right) + EB \right\} \quad (30)$$

¹⁷Wied. Annalen, 21, p. 350; 1884.

¹⁸Rosa, this Bulletin, p. 342, (20).

where F and E are the complete elliptic integrals to modulus $\sin \gamma$, (as in equation 1) and

$$A = \frac{\cos^2 \gamma}{12d^2} \left(a_1 - a_2 - a_3 + (2a_2 - 3a_3) \cos^2 \gamma + 8a_3 \cos^4 \gamma \right)$$

$$B = \frac{\sin^2 \gamma}{12d^2} \left(a_1 + \frac{a_2}{2} + 2a_3 + (2a_2 + 3a_3) \cos^2 \gamma + 8a_3 \cos^4 \gamma \right)$$

The values of a_1 , a_2 , and a_3 are as follows:

$a_1 = b^2 - c^2 + \frac{c^4}{30a^2}$	For square section: $a_1 = \frac{b^4}{30a^2}$
$a_2 = \frac{5b^2c^2 - 4c^4}{60a^2}$	" " " $a_2 = \frac{b^4}{60a^2}$
$a_3 = \frac{2b^4 + 2c^4 - 5b^2c^2}{20d^2}$	" " " $a_3 = -\frac{b^4}{20d^2}$

This formula (30) is a very exact formula for all positions of the two coils, except when they are quite close together. We give later illustrations and tests of the formula and a comparison of its results with the results of other formulæ.

USE OF FORMULÆ FOR SELF-INDUCTANCE IN CALCULATING MUTUAL INDUCTANCE.

One can obtain the mutual inductance of adjacent coils, or of coils at a distance from one another, by means of a formula for the self-inductance of coils. Thus, suppose we have a coil of rectangular section, which we subdivide into three equal parts, 1, 2, 3, Fig. 9. Let L be the self-inductance of the whole coil, L_1 be the self-inductance of any one of the three equal smaller coils, and L_2 be the self-inductance of two adjacent coils taken together. Also let M_{12} be the mutual inductance of coil 1 on coil 2, or of coil 2 on coil 3, and M_{13} be the mutual inductance of coil 1 on coil 3. Then,

$$L = 3L_1 + 4M_{12} + M_{13}$$

$$\text{Also, } L_2 = 2L_1 + 2M_{12}$$

$$\left. \begin{aligned} \therefore M_{12} &= \frac{L_2 - 2L_1}{2} \\ M_{13} &= \frac{L - L_1 - 2L_2}{2} \end{aligned} \right\} \quad (31)$$

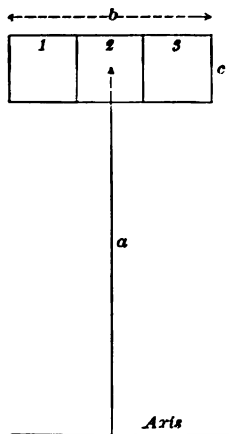


Fig. 9

Formula (31) will thus enable us to find the mutual inductance of two coils of equal radii adjacent or near each other by the calculation of self-inductances from such formulæ as those of Weinstein and Stefan. These latter formulæ are not, however, exact enough when the section is large to permit us to apply it to coils at any considerable distance from one another.

GEOMETRIC MEAN DISTANCE FORMULA.

The mutual inductance of two coaxial coils adjacent or very near can sometimes be obtained by means of the geometric mean distances. This method is accurate only when the sections are very small relatively to the radius. It can often be used to advantage in testing other formulæ, but not often in determining the mutual inductance of actual coils.

Formula (8) gives the mutual inductance of two very near coaxial coils in terms of the geometric mean distance, if r be replaced by R , the geometric mean distance of the two sections. Formula (8) gives M_0 if r be used, where r is the distance between centers. Thus,

$$\frac{\Delta M}{\pi n^2} = 4a \cdot \log \frac{r}{R} \left(1 + \frac{c}{2a} \right) \quad (32)$$

For coils A and C, $R < r$ and ΔM is positive.

" " A " B, $R > r$ and ΔM is negative.

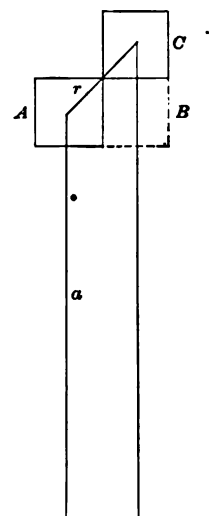


Fig. 10

FORMULÆ AND EXAMPLES.

FORMULÆ FOR CIRCLES.

1. Maxwell. Absolute, in elliptic integrals, for all cases.
2. Maxwell. Absolute (derived from 1 by Landen's transformation), especially for circles near together.
3. Series for F , complete elliptic integral of first kind.

4. Series for E , complete elliptic integral of second kind.
These formulæ are especially convenient for circles close together.
5. Weinstein. Series for M , using complementary modulus k' ; d small.
6. Nagaoka. Using q -series and k' ; d large.
7. Nagaoka. Using q -series and k ; d small.
8. Maxwell. Algebraic series, d small, radii unequal.
9. Maxwell. Algebraic series, d small, radii equal.
10. Coffin. Extension of (9), radii equal.
11. Extension of (8), radii unequal.
12. For coplanar coaxial circles.

FORMULÆ FOR COILS OF RECTANGULAR SECTION.

13. Rowland. Coils of equal radii but unequal section.
14. Rowland. Coils of equal radii and equal section.
15. Expressions for $\frac{d^2 M_0}{dx^2}$ and $\frac{d^2 M_0}{da^2}$.
16. Differential equation for mutual inductance of two coaxial coils.
17. Differential equation for mutual inductance of two coaxial coils of equal section and equal radii.
18. Rayleigh. Formula of quadratures. M as mean of 9 values. (Radii may be unequal, but distance must not be small.)
19. Rayleigh. Same for coils of equal radii and equal section.
20. Expression for error of Rayleigh's formula.
21. Expression for error of Rowland's formula.
22. Lyle. Using an equivalent radius, square section.
(Radii may be unequal, but distance must not be too small.)
23. Lyle. Same for coils of sections not square.
24. Expression for ΔM based on Lyle's theorem, radii equal; sections square and equal.

- 24a. Expression for ΔM based on Lyle's theorem, radii unequal; sections square and equal.
25. Stefan. Algebraic series for M , radii equal. (Error in formula.)
26. Rosa. Formula for ΔM , radii equal, rectangular section.
27. Rosa. Formula for ΔM , radii equal, square section.
28. Rosa. Formula for ΔM , radii equal, simple approximate formula, square section.
29. Weinstein. Formula for M , using elliptic integrals. (Small error in formula.)
30. Rosa-Weinstein. Revised formula, expression for ΔM .
31. Formula for M in terms of self-inductances.
32. Formula for M by geometric mean distance.

ILLUSTRATION AND TESTS OF THE FORMULÆ.

COAXIAL CIRCLES.

Formula 1.—Maxwell's. For any two coaxial circles.

Example 1: Let $a = A = 25$ cm
 $d = 20$ cm.

$$k = \frac{50}{\sqrt{2500 + 400}} = 0.9284766 = \sin \gamma$$

$$\gamma = 68^\circ 11' 54''.88 = 68^\circ 198578.$$

From Legendre's tables, we obtain

$$\log F = 0.3852191$$

$$\log E = 0.0547850$$

$$\frac{2}{k} - k = 1.2255892 \quad \left(\frac{2}{k} - k\right) F = 2.9755281$$

$$\frac{2}{k} = 2.1540658, \quad -\frac{2}{k} E = -2.4436781$$

$$\left(\frac{2}{k} - k\right) F - \frac{2}{k} E = 0.5318500$$

$$4a = 100$$

$$\therefore \frac{M}{\pi} = 53.1850 \text{ cm}$$

Using Maxwell's tables (given in Maxwell, Appendix II, Vol. II)

$$\log_{10} \frac{M}{4\pi a} \text{ for } 68^\circ 1 = 1.7230640$$

$$\text{for } 68^\circ 2 = 1.7258286$$

for $68^\circ 198578 = 1.7257893$ by simple interpolation.

$$\therefore \frac{M}{4\pi a} = 0.531850,$$

$$\frac{M}{\pi} = 53.1850 \text{ cm, as above.}$$

The calculation of mutual inductance by the above methods is simplest for circles not near each other, as then the values of $\log F$, $\log E$, and $\log \frac{M}{4\pi\sqrt{Aa}}$ are very exact when taken by simple interpolation. When γ is nearly 90° , however, second and third differences have to be used in interpolation.

Formula 2.—Maxwell's second expression. For circles near each other.

Example 2: Let $a = A = 25 \text{ cm}$

$$d = 1 \text{ cm}$$

$$\text{In this case } k = \sin \gamma = \frac{50}{\sqrt{2501}} = .99980006$$

$$\gamma = 88^\circ 51' 14''$$

This value of γ is so nearly 90° that it is difficult to obtain accurate values of F and E from tables of elliptic integrals, or of $\frac{M}{4\pi a}$ from Maxwell's table.

We may therefore use formula (2) instead of (1).

$$r_1 = \sqrt{2501} = 50.01 \text{ nearly, } r_2 = 1.0$$

$$\therefore k_1 = \sin \gamma_1 = \frac{r_1 - r_2}{r_1 + r_2} = \frac{49.01}{51.01} = 0.960792$$

$$\gamma_1 = 73^\circ 54' 9.''7 = 73^\circ 9027$$

From Legendre's tables, for $\gamma_1 = 73^\circ 9' 27''$, $F_1 = 2.7024553$

$$E_1 = 1.0852167$$

$$F_1 - E_1 = 1.6172386$$

$$\frac{8\sqrt{Aa}}{\sqrt{k_1}} = \frac{200}{\sqrt{.960792}} \therefore \frac{8\sqrt{Aa}}{k_1} (F_1 - E_1) = \frac{M}{\pi} = 329.9814 \text{ cm.}$$

Formulæ 3 and 4.—Series for F and E.

Suppose that, in the last example, we calculate F and E by means of formulæ 3 and 4, instead of taking them from Legendre's tables.

Example 3: $A = a = 25$, $d = 1$.

First for F : $k'^2 = 1 - k^2 = 1 - \frac{2500}{2501} = \frac{1}{2501}$

$$\log \frac{4}{k'} = \frac{1}{2} \log \frac{16}{k'^2} = \frac{1}{2} \log_e 40016 = .000200 + \log_e 200 = 5.298517$$

$$\frac{1}{4} k'^2 \left(\log \frac{4}{k'} - 1 \right) = .000430$$

$$F = 5.298947$$

Second for E :

$$1 + \frac{k'^2}{2} \left(\log \frac{4}{k'} - \frac{1}{2} \right) = 1.000960$$

$$\frac{3k'^4}{16} \left(\log \frac{4}{k'} - \frac{13}{12} \right) = .000000$$

$$\therefore E = 1.000960$$

If these values of F and E be substituted in formula (1), k being 0.9998002, we obtain the same value of M as by formula (2).

Example 4: $A = 25$, $a = 20$, $d = 10$ cm. (See Fig. 1.)

$$k^2 = \frac{4 \times 20 \times 25}{(45)^2 + (10)^2} = \frac{16}{17} \therefore k'^2 = \frac{1}{17}$$

$$\log \frac{4}{k'} = \frac{1}{2} \log (16 \times 17) = \frac{1}{2} \log_e 272 = 2.8029010$$

$$\frac{k'^2}{4} \left(\log \frac{4}{k'} - 1 \right) = .0265132$$

$$\frac{9k'^4}{64} \left(\log \frac{4}{k'} - \frac{7}{6} \right) = .0007962$$

$$\begin{aligned}
 \frac{25k'^8}{256} \left(\log \frac{4}{k'} - \frac{111}{90} \right) &= .0000312 \\
 \frac{1225k'^8}{16384} \left(\log \frac{4}{k'} - 1.27 \right) &= .0000014 \\
 \therefore F &= 2.8302430 \\
 1 + \frac{k'^2}{2} \left(\log \frac{4}{k'} - \frac{1}{2} \right) &= 1.0677324 \\
 \frac{3k'^4}{16} \left(\log \frac{4}{k'} - \frac{13}{12} \right) &= .0011156 \\
 \frac{15k'^4}{128} \left(\log \frac{4}{k'} - 1.20 \right) &= .0000381 \\
 \frac{175k'^8}{2048} \left(\log \frac{4}{k'} - 1.25 \right) &= .0000017 \\
 \therefore E &= 1.0688878
 \end{aligned}$$

To find the value of M we now use equation (1).

$$\begin{aligned}
 k &= \sqrt{\frac{16}{17}} = 0.9701425 \\
 \frac{2}{k} - k &= 1.0914105 \\
 \left(\frac{2}{k} - k \right) F &= 3.088957 \\
 \frac{2}{k} \cdot E &= 2.203569 \\
 \text{Difference} &= \left\{ \left(\frac{2}{k} - k \right) F - \frac{2}{k} E \right\} = 0.885388 \\
 \text{Multiplying by } 4\sqrt{Aa} &= 4\sqrt{500} \text{ gives} \\
 \frac{M}{\pi} &= 79.19150 \text{ cm.}
 \end{aligned}$$

Formula 5.—Weinstein. For any coaxial circles; series more convergent for circles not far apart.

Example 5: $A=30$, $a=25$, $d=0$.

Two coplanar circles.

$$k^2 = \frac{4 \times 30 \times 25}{(30+25)^2} = \frac{120}{121}$$

$$\begin{aligned}
 k'^2 &= \frac{1}{121} & k' &= \frac{1}{11}, \frac{4}{k'} = 44 \\
 \log_e 44 - 1 &= 2.784190 \\
 1 + \frac{3k'^2}{4} + \frac{33k'^4}{64} + \frac{107k'^6}{256} &= 1.006234 \\
 \text{Product} &= 2.801546 \\
 1 + \frac{15k'^4}{128} + \frac{185k'^6}{1536} &= 1.000008 \\
 \text{Difference} &= 1.801538
 \end{aligned}$$

$$4\pi\sqrt{Aa} = 4\pi a\sqrt{1.2}$$

$$\therefore \frac{M}{\pi} = 197.3485 \text{ cm}$$

Example 6: Take the same circles as in example 4.

$$A = 25, \quad a = 20, \quad c = 5, \quad d = 10.$$

$$k'^2 = \frac{1}{17}, \quad \log \frac{4}{k'} - 1 = 1.802901$$

$$1 + \frac{3k'^2}{4} = 1.0441176$$

$$1 + \frac{15}{128}k'^4 = 1.0004053$$

$$\frac{33}{64}k'^4 = .0017842$$

$$\frac{185}{1536}k'^6 = .0000245$$

$$\frac{107}{256}k'^6 = .0000851$$

$$\frac{7465}{65536}k'^8 = .0000012$$

$$\frac{5913}{16384}k'^8 = .0000042$$

$$1.0004310 = C$$

$$\text{Sum} = 1.0459911 = B$$

$$B \log \left(\frac{4}{k'} - 1 \right) = 1.8858184$$

$$\left\{ B \log \left(\frac{4}{k'} - 1 \right) - C \right\} = 0.8853874$$

Multiplying by $4\sqrt{500}$ gives $\frac{M}{\pi} = 79.1915 \text{ cm}$, agreeing with value found by other method.

Formula 6.—Nagaoka. Using q -series, for circles not near each other.

Example 7: $A=a=25$, $d=20$

$$\sqrt{k'} = \sqrt{\cos \gamma} = \left(\frac{20}{\sqrt{2900}} \right)^{\frac{1}{2}} = 0.6094183$$

$$\frac{l}{2} = \frac{1 - \sqrt{k'}}{2(1 + \sqrt{k'})} = \frac{1}{2} \frac{0.3905817}{1.6094183} = 0.1213425$$

$$2 \left(\frac{l}{2} \right)^5 = .0000526$$

$$15 \left(\frac{l}{2} \right)^9 = .0000000$$

$$\therefore q = 0.1213951$$

$$3q^4 = + .0006516$$

$$-4q^6 = - .0000128$$

$$+9q^8 = + .0000004$$

$$\epsilon = .0006392$$

$$1 + \epsilon = 1.0006392$$

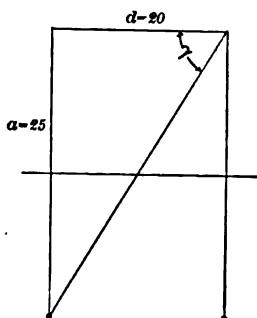


Fig. 11

$$\log (1 + \epsilon) = 0.0002773$$

$$\log q^{\frac{1}{2}} = 2.6263018$$

$$\log 16\pi\sqrt{Aa} = 3.0992099$$

$$\log \frac{M}{\pi} = 1.7257890 \therefore \frac{M}{\pi} = 53.1850 \text{ cms, as found by formula 1, Example 1.}$$

Formula 7.—Nagaoka. Using q -series, for coaxial circles near each other.

Example 8: $A=a=25$, $d=4$

$$k = \sin \gamma = \frac{50}{\sqrt{2516}} \sqrt{k} = .99840637$$

$$l_1 = \frac{1 - \sqrt{k}}{1 + \sqrt{k}} = \frac{.00159363}{1.9984064}, \frac{l_1}{2} = .00039872 = q,$$

as $\left(\frac{l_1}{2}\right)^5$ and higher powers can be neglected.

$$\frac{1}{q_1} = 2508.04, \log_e \left(\frac{1}{q_1} \right) = 7.827238$$

$$(1 - q_1 + 4q_1^3) = 0.9996019$$

$$8q_1 = 0.00318976$$

$$\left\{ \log_e \left(\frac{1}{q_1} \right) [1 + 8q_1(1 - q_1 + 4q_1^3)] - 4 \right\} = 3.852195 = A$$

$$\frac{1}{2(1 - 2q_1)^2} = 0.5007985 = B$$

$$4\sqrt{Aa} = \frac{100}{} = C$$

$$\text{Product } A \times B \times C, = M/\pi, = 192.9174 \text{ cm.}$$

There is a difficulty in using the above formula owing to the fact that when k is nearly unity the numerator of the expression for l_1 is small, and unless the value of k is carried out to about eight decimal places the value of M may be appreciably in error. For approximate calculations a seven-place table of logarithms is sufficient, and it is not very troublesome to carry out this one number to the necessary number of places for precision calculations. Or, k can easily be computed to any degree of accuracy without logarithms. The same thing applies to formulæ 3 and 4, where k' must be computed with great precision when it is quite small.

Formula 8.—Maxwell. Two circles of unequal radii. Formula accurate only for circles very near each other.

Example 9: $A = 26$, $a = 25$, $d = 1$, $c = 1$ and $r = \sqrt{2}$

$$\text{Since } r = \sqrt{2}, \log_e \frac{8a}{r} = \log \frac{200}{\sqrt{2}} = 4.951744$$

$$1 + \frac{c}{2a} = 1.020000 \quad 2 + \frac{c}{2a} = 2.020000$$

$$\frac{c^2 + 3a^2}{16a^2} = .000400 \quad - \frac{3c^2 - d^2}{16a^2} = -.000200$$

$$- \frac{c^3 + 3ca^2}{32a^3} = -.000008 \quad - \frac{c^3 - 6ca^2}{48a^3} = -.000010$$

$$1.020392 = B$$

$$2.019790 = C$$

$$\begin{aligned}
 B \log \frac{8a}{r} &= 5.052310 \\
 C &= 2.019790 \\
 \left\{ B \log \frac{8a}{r} - C \right\} &= 3.032520 \quad \text{Multiply by } 4a=100 \text{ and} \\
 \frac{M}{\pi} &= 303.252 \text{ cm.}
 \end{aligned}$$

This formula would be less accurate for the circles of problem 4, but is accurate for circles close together, as this problem shows.

Formula 9.—Maxwell. Circles of equal radii near each other.

Example 10: $A=a=25$, $d=1$

$$\begin{aligned}
 \frac{8a}{d} &= 200 \quad \log_e 200 = 5.298317 \\
 \log \frac{8a}{d} \left(1 + \frac{3d^2}{16a^2} \right) &= 1.000300 \times 5.298317 = 5.29990 \\
 \left(2 + \frac{d^2}{16a^2} \right) &= \frac{2.00010}{3.29980} \\
 \text{Multiply by } 4a=100 & \\
 \frac{M}{\pi} &= 329.980,
 \end{aligned}$$

nearly agreeing with the more exact value found under problem 2.

This is a very simple and convenient formula for equal circles, and gives approximate results for circles still farther apart than in this problem.

Formula 10.—Coffin. Circles of equal radii, not farther apart than $d=a$.

Example 11: $A=a=25$, $d=16$

$$\begin{aligned}
 \frac{8a}{d} &= 12.5 \quad \log_e 12.5 = 2.5257286 \\
 \text{First series of terms} &= B = 1.074478 \\
 \text{Second series of terms} &= C = 2.023220 \\
 \therefore \left\{ B \log \frac{8a}{d} - C \right\} &= 0.690620 \\
 4a &= 100 \\
 \therefore \frac{M}{\pi} &= 69.0620
 \end{aligned}$$

This agrees with the value given by formula 1 within one part in 200,000. As the distance apart of the circles increases the accuracy by this formula of course gradually decreases.

Formula 11.—Extension of Maxwell's series formula for unequal circles.

Example 12: $A=25$, $a=20$, $c=5$, $d=10$

$$r = \sqrt{c^2 + d^2} = 5\sqrt{5}, \quad \log \frac{8a}{r} = \log \frac{32}{\sqrt{5}} = 2.6610169$$

$$-\left(2 + \frac{c}{2a}\right) = -2.125000$$

$$-\frac{d^2 - 3c^2}{16a^2} = -.003906$$

$$+ \frac{93d^4 - 534c^2d^2 - 19c^4}{6144a^4} = .000424$$

$$- \frac{1845cd^4 - 3030c^3d^2 - 379c^5}{61440a^5} = -.000271$$

$$+ \frac{6cd^3 - c^3}{48a^3} = +.007487$$

$$C = -2.122114$$

$$1 + \frac{c}{2a} = 1.125000$$

$$\frac{c_2 + 3d^2}{16a^2} = +0.050781$$

$$\frac{45cd^4 - 30c^3d^2 - 19c^5}{2048a^5} = +0.000277$$

$$\frac{3cd^3 + c^3}{32a^3} = -0.006348$$

$$\frac{15d^4 - 42c^2d^2 - 17c^4}{1024a^4} = -0.000210$$

$$B = 1.169500$$

$$B \log_e \frac{8a}{r} = 3.112060$$

$$C = 1.122114$$

$$0.989946$$

multiplying by $4a=80$, $\frac{M}{\pi} = 79.1957 \text{ cm}$

This result is correct to one part in 19,000 (see examples 4 and 6). Using only the first three terms for B and C , the result would be too large by one part in 1750.

Formula 12.—Extension of Maxwell's series for coplanar circles.

Example 13: $A = 25$ $a = 20$ $c = 5$ $d = 0$

$$\begin{array}{rcl}
 1 + \frac{c}{2a} = & 1.125000 & - \frac{c^2}{32a^3} = -0.000488 \\
 \frac{c^2}{16a^3} = & 0.003906 & - \frac{19c^5}{2048a^5} = -0.000009 \\
 \frac{17c^4}{1024a^4} = & 0.000065 & B' = -0.000497 \\
 & + 1.128971 & - \left(2 + \frac{c}{2a}\right) = -2.125000 \\
 B' = & -0.000497 & - \frac{c^3}{48a^3} = -0.000326 \\
 B = & 1.128474 & - \frac{19c^4}{6144a^4} = -0.000012 \\
 \log \frac{8a}{c} = & 3.4657359 & - 2.125338 \\
 B \log \frac{8a}{c} = & 3.910994 & \frac{3c^8}{16a^8} = +0.011719 \\
 C = & -2.113613 & \\
 & 1.797381 & \\
 \text{Multiply } 4a = & 80 & \frac{379c^5}{61440a^5} = C = +0.000006 \\
 \frac{M}{\pi} = & 143.79048 & \\
 \end{array}$$

MUTUAL INDUCTANCE OF COILS OF RECTANGULAR SECTIONS.

Formulae 13, 14, 15.—Rowland. Coils of equal radii not very near each other.

Example 14: $A = a = 25$, $b = c = 2$ cm, $d = 10$.

The mutual inductance of the two coils is $M = M_0 + \Delta M$.

We find M_0 by formula 1, 6, or 10, and ΔM by 14 and 15.

$$M_0 = 107.4885\pi$$

$$k = \sin \gamma = \frac{50}{\sqrt{2600}} = 0.9805807$$

$$k^2 = .9615383$$

$$\log_{10} F = 0.4821754$$

$$\log_{10} E = 0.0207625$$

Substituting these values in formula (15) we obtain

$$\frac{1}{\pi} \frac{d^2 M}{da^2} = -0.9081$$

$$\frac{1}{\pi} \frac{d^2 M}{dx^2} = +1.0639 \quad b^2 = c^2 = 4.$$

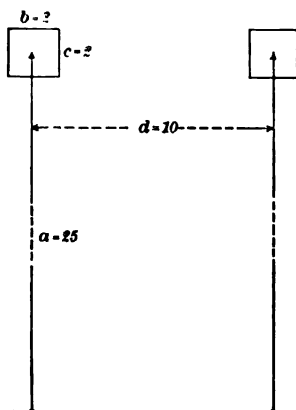


Fig. 12

Substituting these values in formula (14) we obtain

$$\Delta M = .05193 \pi$$

$$\therefore M = M_0 + \Delta M = (107.4885 + .0519) \pi$$

$$= 107.5404 \pi \text{ cm.}$$

The correction ΔM thus amounts to about one part in 2000 of M . At a distance $d=20$ cm; the correction is over one part in 1000. For a coil of section 4×4 cm at $d=10$, ΔM would be four times as large as the value above, or about 1 part in 500, and at 20 cm 1 part in 250.

Formulae 18 and 19.—Rayleigh. Coils of equal or unequal radii, not very near each other.

Example 14a: $A=a=25$, $b=4$, $c=1$, $d=10$

We now find by formula (1) in accordance with formula (19) the mutual inductance of the following pairs of circles: O, 1 when $a=25$, $A=25.5$, $d=10$.

O, 4 when $a=25$, $A=24.5$, $d=10$; O, 2 when $a=A=25$ and $d=8$; O, 3 when $A=a=25$, $d=12$, and finally O, O' when $A=a=25$, $d=10$. Thus:

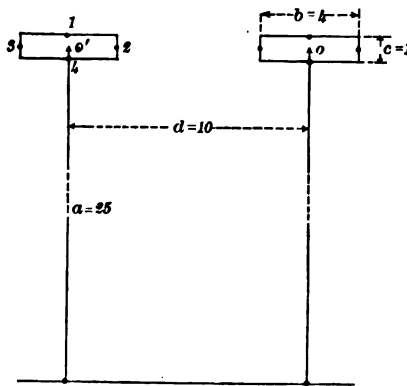


Fig. 18

$$M_1 = 109.3217\pi$$

$$M_4 = 105.4287\pi$$

$$M_2 = 127.3949\pi$$

$$M_3 = 91.9206\pi$$

$$\hline 434.0659\pi$$

$$M_0 = 107.4885\pi$$

$$\hline 326.5774\pi$$

$$\therefore M = 108.8591\pi$$

$$M_0 = 107.4885\pi$$

$$\hline \frac{\Delta M}{\pi} = 1.3706 \text{ cm.}$$

Formulae 22 and 23.—Lyle. For coils of square section of equal or unequal radii, not very near one another; also for coils of rectangular section not square.

The equivalent radius $r = a \left(1 + \frac{b^2}{24a^2} \right)$

Example 15: $A=a=25$ cm, $b=c=2$ cm, $d=10$ cm.

$$r = 25 \left(1 + \frac{4}{15000} \right) = 25.00667 \text{ cm.}$$

M is now found by using formula 1, 6, or 10, employing r in place of a as the radius.

The result is $M=107.5402$, agreeing very closely with the result found under example 14.

$$M - M_0 = \Delta M = .0517.$$

Example 16: $A=a=25$, $b=4$, $c=1$, $d=10$

$$r = 25 \left(1 + \frac{1}{15000} \right) = 25.00167$$

$$\beta^2 = \frac{b^2 - c^2}{12} = \frac{15}{12} = 1.25, \quad 2\beta = 2.236 \text{ cm, the distance apart of the}$$

two filaments which replace the coil. We now find by formula 1, 6, or 10 the mutual inductances of two circles 1, 2 on the two circles 3, 4, where $a = 25.00167$ and d is 7.764, 10 and 12.236 cm, respectively. Thus:

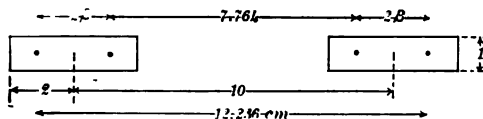


Fig. 14

$$2 M_{13} = 215.00228\pi$$

$$M_{14} = 90.31304\pi$$

$$M_{23} = 130.14060\pi$$

$$4 M = 435.45592\pi$$

$$\therefore M = 108.8640 \pi$$

$$M_0 = 107.4885 \pi$$

$$\frac{\Delta M}{\pi} = 1.3755$$

ΔM = the correction for section of the coils whose dimensions are given above. These values of M and ΔM agree nearly with the results obtained in Example 14a above.

Formula 24.—For ΔM . Radii of coils equal, square section.

Example 17: $A = a = 25$, $d = 10$, $b = c = 2$

$$\log \frac{8a}{d} - 1 = 1.9957$$

$$- \frac{3a^2}{16a^2} \left(\log \frac{8a}{d} - \frac{4}{3} \right) = - \frac{0.0499}{1.9458}$$

$$\frac{b^2}{6a} = \frac{4}{150}$$

$$\therefore \frac{\Delta M}{\pi} = .0514 \text{ agreeing closely with the}$$

result in Example 15.

Formula 24a.—For ΔM . Radii of the coils unequal, section square.

Example 18: $A = 25$, $a = 20$, $d = 6$, $b = 2$ (section 2 x 2 cm)
Substituting in formula 24a we obtain

$$\frac{\Delta M}{\pi} = .0740$$

Using Lyle's method of computing M we obtain

$$\begin{aligned} M &= 106.930 \\ M_0 &= 106.855 \\ \Delta M &= .075 \end{aligned}$$

This difference amounts to one part in 100,000 of the value of M . Formula (24a) is less convenient than (22) when one wants only the value of M . It is, however, valuable as a check when the coils are not far apart. It is not reliable for coils at a distance.

Formula 25.—Stefan. For coils of equal radii, and rectangular section.

This formula has been shown to be in error, but is substantially right for coils near together. We give in Fig. 15 the results of calculations for coils 2 x 2 cm section at different distances, showing how much formula 25 is in error. No numerical example will be given here as this formula should not be employed. Figure 15 shows the magnitude of the error in this formula for one particular pair of coils.

Formula 26.—Rosa. Coils of equal radii. Most accurate of all the formulæ for coils near together (but not in contact), and less accurate when coils are far apart.

Example 19: $A = a = 25$, $b = 4$, $c = 1$, $d = 10$
(same coils as examples 14a, 16).

$$\log \frac{8a}{d} = \log_e 20 = 2.9957$$

$$\frac{3b^2 + c^2}{96a^2} \cdot \log \frac{8a}{d} = \frac{49 \times 2.9957}{60,000} = .0024465$$

$$\frac{b^2 - c^2}{12d^2} = \frac{15}{1200} = .0125000$$

$$\frac{2b^4 + 2c^4 - 5b^2c^2}{120d^4} = \frac{434}{1,200,000} = .0003617$$



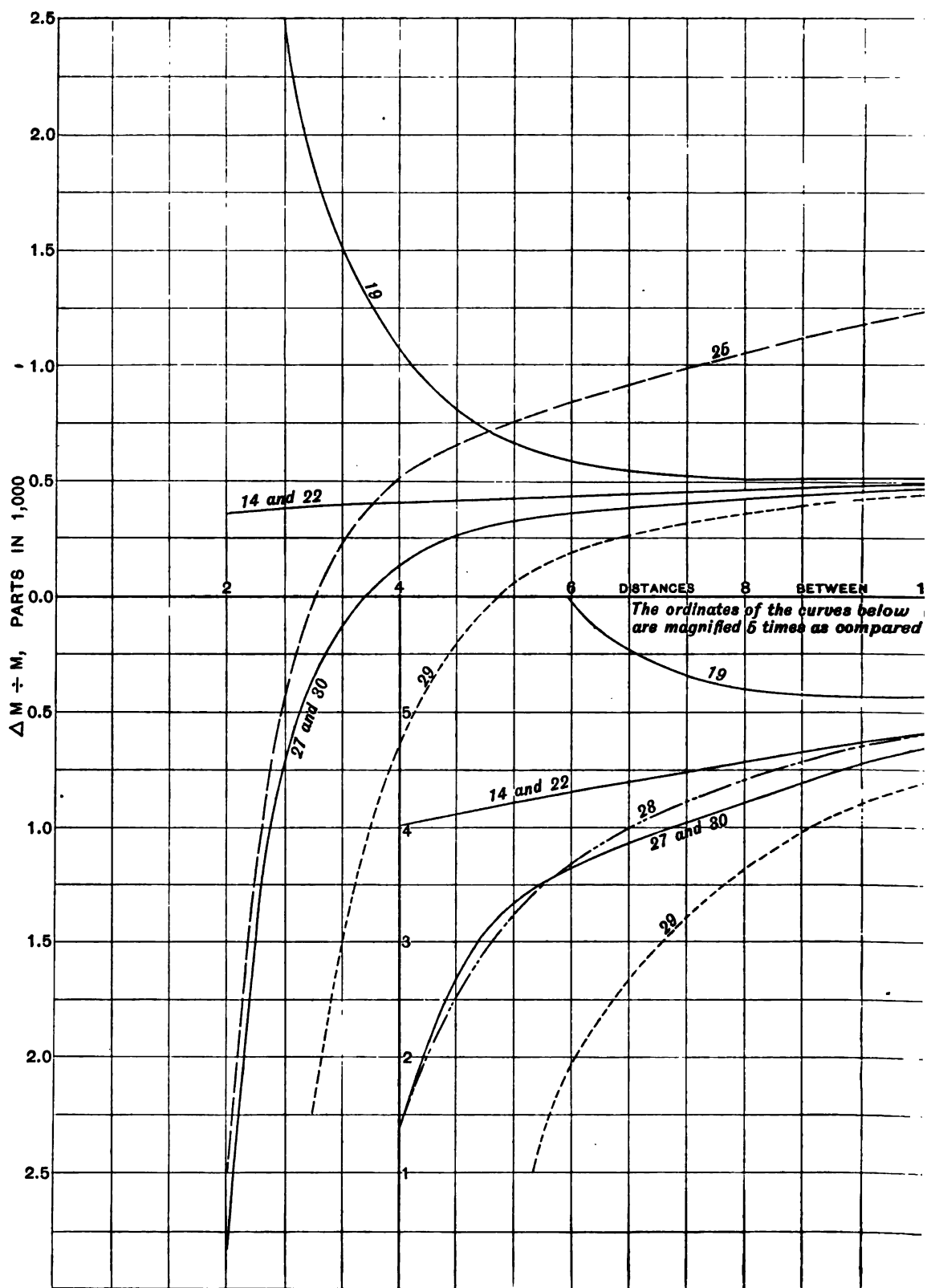
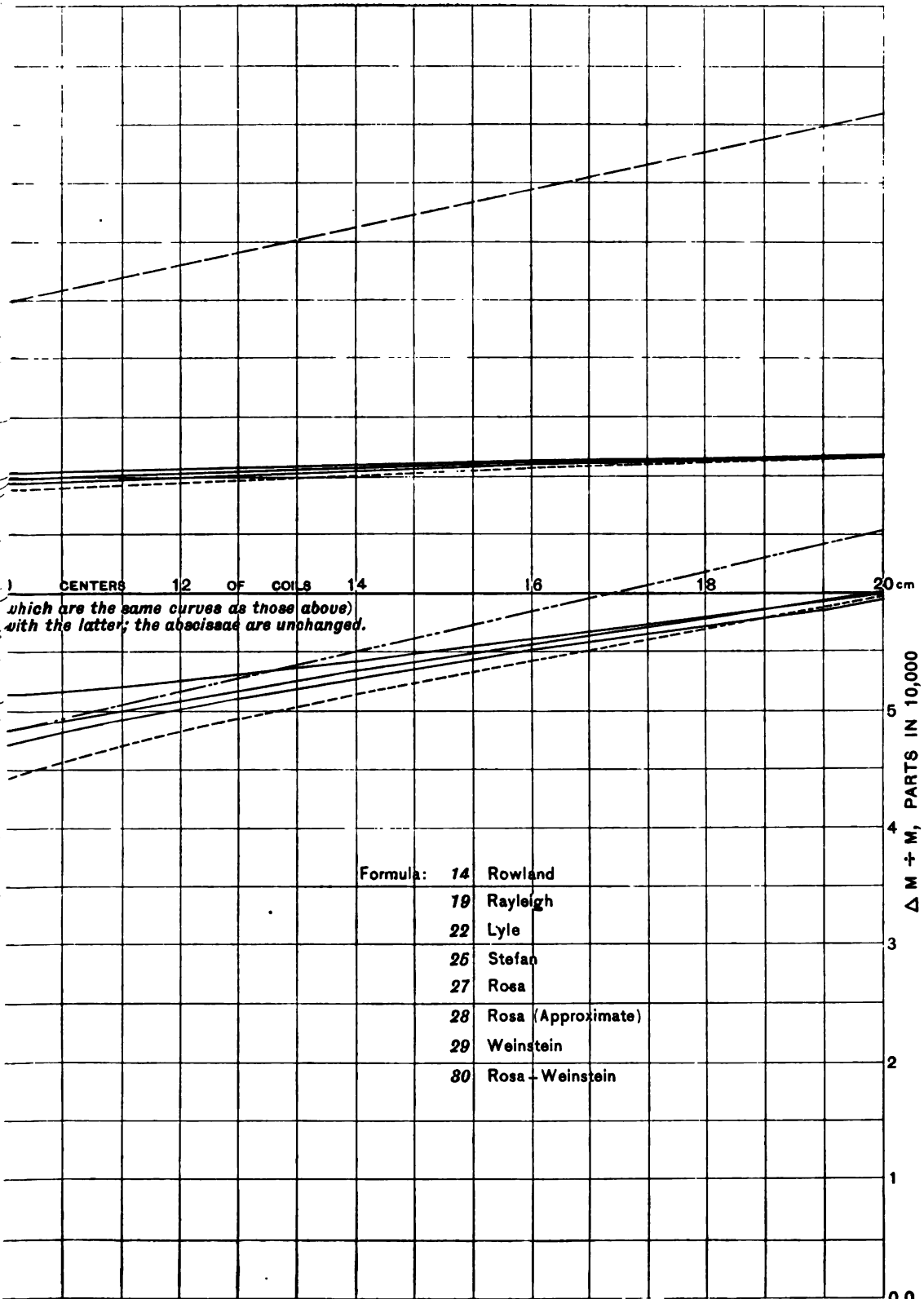
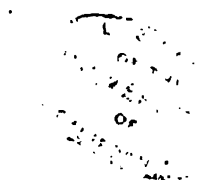


Fig. 15.—Comparison of Formulae for a pair of coils of equal radii (25 cm) and equal section (2x2 cm) at varying distances (



2 to 20 cm, center to center). Corrections $\frac{\Delta M}{M}$ are negative below the axis. The curve representing Formulae 27 and 30, one part in 100,000 of M.



$$\begin{aligned}
\frac{3b^4 - 3c^4 + 14b^2c^2 - 14b^4c^2}{504d^8} &= \frac{8925}{504 \times 10^8} = .0000177 \\
\frac{6b^4 + 6c^4 + 5b^2c^2}{5760a^2d^4} &= \frac{1622}{360 \times 10^8} = .0000045 \\
\frac{7c^2d^2}{1024a^4} \left(\log \frac{8a}{d} - \frac{163}{84} \right) &= .0000018 = .0153322 \\
-\frac{11b^2 - 3c^2}{192a^2} &= -\frac{173}{120,000} = -.0014417 \\
-\frac{15b^2d^2}{1024a^4} \left(\log \frac{8a}{d} - \frac{97}{60} \right) &= -.0000827 = -.0015244 \\
&\quad .0138078 \\
4a = 100 \therefore \frac{4M}{\pi} &= 1.3808 \text{ cm.}
\end{aligned}$$

This is a little larger value than found by formula 19 and 23, and we shall see later that it is more nearly correct than either of the other values.

Formulae 27 and 28.—Rosa. Coils of equal radii and square section.

Example 20: $A = a = 25, \quad b = c = 2, \quad d = 10$

$$\begin{aligned}
\log \frac{8a}{d} - 1 &= 2.9957 - 1 = 1.9957 \\
\frac{17b^2}{240d^2} &= \frac{68}{24,000} = .0028 = 1.9985 \\
-\frac{a^2b^2}{5d^4} &= -\frac{2500}{50,000} = -.0500 \\
-\frac{3a^2}{16a^4} \left(\log \frac{8a}{d} - 4 \right) &= \frac{300 \times 1.6624}{10,000} = -.0499 = -.0999 \\
&\quad 1.8986 \\
\frac{b^2}{6a} = \frac{4}{150} \therefore \frac{4M}{\pi} &= .05063
\end{aligned}$$

The approximate formula (28) would have given .0519 (agreeing with formulæ 14 and 22), which would be amply accurate for any experimental purpose. When the section is larger these small terms are, however, more important.

Example 21: $A=a=25$, $b=c=5$, $d=10$

$$\begin{aligned}\log \frac{8a}{d} - 1 &= 1.9957 \\ \frac{17b^3}{240d^3} &= .0177 = 2.0134 \cdot \\ \frac{-a^3b^3}{5d^4} &= - .3125 \\ -\frac{3a^2}{16a^3} \left(\log \frac{8a}{d} - \frac{4}{3} \right) &= - \frac{.0499}{1.6510} = - .3624 \\ \frac{b^3}{6a} &= \frac{25}{150} \\ \therefore \frac{\Delta M}{\pi} &= 0.2752 \\ M_0 &= 107.4885 \text{ (see example 14.)} \\ M &= 107.7637 \text{ cm.}\end{aligned}$$

This is a very simple formula for computing ΔM , and within a considerable range (i. e., d not larger than a and yet the coils not in contact) it is very accurate.

Formula 29.—Weinstein. Formula for mutual inductance of equal coils, in elliptic integrals.

This formula is very accurate for coils not near together, but is much less accurate for coils relatively near. See Fig. 15. The revised form of this formula (30) is more accurate as well as more convenient. and hence no example will be given of (29).

Formula 30.—Rosa-Weinstein. For ΔM , for two coils of equal radii and equal section.

Example 22: $A=a=25$, $b=4$, $c=1$, $d=10$

$$\begin{aligned}a_1 &= 15.0000533 & \sin^2 \gamma &= \frac{2500}{2600} = \frac{25}{26} \\ a_2 &= 0.0020267 & \cos^2 \gamma &= \frac{100}{2600} = \frac{1}{26} \\ a_3 &= 0.217 & \frac{c^2}{24a^2} &= .0000667\end{aligned}$$

$$a_1 - a_2 - a_3 + (2a_2 - 3a_3)\cos^2\gamma + 8a_3\cos^4\gamma = 14.7587120$$

$$a_1 + \frac{a_2}{2} + 2a_3 + (2a_2 + 3a_3)\cos^2\gamma + 8a_3\cos^4\gamma = 15.4628292$$

$$A = 0.0004730 \quad \text{Also } F = 3.0351168$$

$$B = 0.0123901 \quad E = 1.0489686$$

$$(F - E) \left(A + \frac{c^2}{24a^2} \right) = 0.0010719$$

$$\dot{E}B = \underline{0.0129968}$$

$$\text{Sum} = 0.0140687$$

$$4a \sin \gamma = 100 \sqrt{\frac{25}{26}} \therefore \frac{\Delta M}{\pi} = 1.3795 \text{ cm.}$$

This is not as simple to calculate as (26) and when d is less than $a/2$ is less accurate than (26). But for $d = a$ or greater it is more accurate than (26), and indeed the most accurate of all the formulæ.

Formula 31.—Mutual inductance in terms of self-inductance.

This is for coils relatively near. An example will be given below, page 408.

Formula 32.—Mutual inductance by geometric mean distances.

$$\begin{aligned} \text{Example 23: } A &= 25.1 \\ a &= 25.0 \\ b &= c = 0.1 \text{ cm} \\ d &= 0.1 \text{ cm} \end{aligned}$$

The geometrical mean distance of two coils, corner to corner, as in Fig. 10, is 0.997701, and $\log \frac{r}{R} = 0.002302$

$$\begin{aligned} \therefore \frac{\Delta M}{\pi} &= 100 \times 0.002302 (1.002) \\ &= 0.2307. \end{aligned}$$

COMPARISON OF SEVEN FORMULÆ FOR COILS OF SQUARE SECTION.

14, 19, 22, 25, 27, 29, 30.

We have obtained above by different formulæ different values of the correction ΔM . In order to show the degree of divergence among them for a particular pair of coils as the distance between

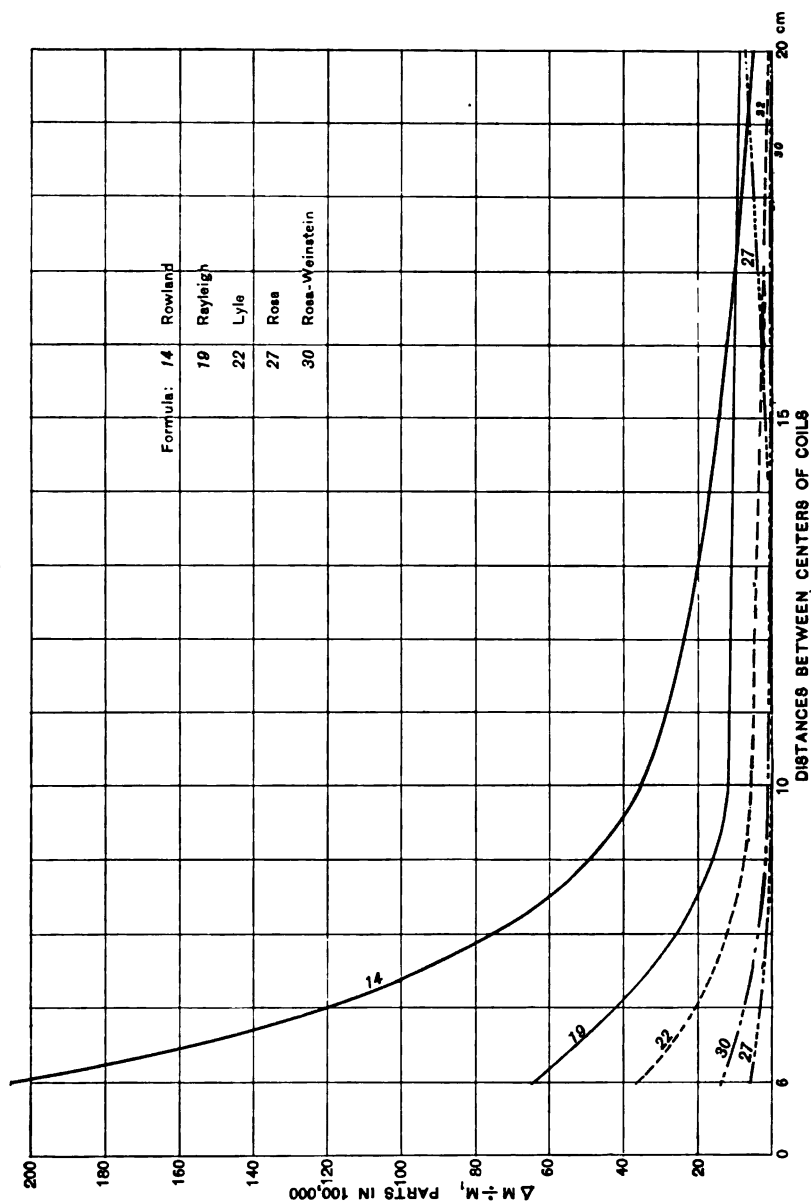


Fig. 16.—Comparison of Formulae for coils of equal radii (25 cm) of equal sections (1×4 cm) at different distances. The ordinates are the errors by the different formulae.

the coils is varied we have made a series of computations by seven different formulæ for two coils of square section, 2×2 cm, and equal radii ($a = 2.5$ cm), for distances varying between 3 and 20 cm. The results are plotted in Figure 15. Formulæ 27 and 30 give the values of $\frac{\Delta M}{M}$ with great precision and agree so closely that they are represented by the same line almost to $d = 20$. ΔM is negative for small distances, and at about $d = 3.6$ cm becomes zero, increasing gradually as the distance increases to 20 cm. ΔM decreases after passing a maximum at about 6 cm, but M decreases faster than ΔM , hence the relative value of the correction is greater at 20 cm than at any point between that and 3 cm. This is true only for coils of square section. As the correction for sixth differentials is zero for a coil of square section, formulæ 27 and 30 agree for small distances, which they do not quite do for coils not of square section. The dotted curves show how formulæ 25 and 29 diverge from the true values given by 27 and 30. Rowland's and Lyle's formulæ (14) and (22) agree almost exactly, both being based on the same assumptions. They are very accurate for coils not near each other. Formula (19) departs farthest from the true values at small distances. The uncertainty of the value of $M = M_0 + \Delta M$ for these coils as given by formulæ 27 and 30 is not as much as one in a hundred thousand.

COMPARISON OF FIVE FORMULÆ FOR COILS NOT OF SQUARE SECTION.

14, 19, 22, 27, 30.

For coils of section not square the corrections are much larger and the formulæ are not so accurate. We have found differences in some of the examples above for the coils of sections 4×1 cm and we will now decompose these coils into four coils each of square section, and calculate ΔM by means of formula (27) which we have seen agrees with (30) and is very accurate. If the two coils A and B be considered to be made up of four coils each, the mutual inductance of A on B will be the sum of the mutual inductances of the four constituents of A on the four separate coils of B. Thus, if M_{ab} be the mutual inductance of A on B and M_{45} that of coil 4 on coil 5, etc., we see that

$$M_{ab} = M_{45} + 2 M_{35} + 3 M_{25} + 4 M_{15} + 3 M_{16} + 2 M_{17} + M_{18}$$

or, $M_{ab} = M_7 + 2 M_8 + 3 M_9 + 4 M_{10} + 3 M_{11} + 2 M_{12} + M_{13} \quad (x)$

where in the second expression M_7 means the mutual inductance of two coils distant 7 cm between centers, as coils 4 and 5 are, etc. Calculating M_0 and ΔM for each of the seven distances from 7 to 13 cm inclusive, and substituting in equation (x) above we shall have M_{ab} the mutual inductance of the two coils to a very high degree of accuracy. These values are given in the accompanying table :

	$\frac{M}{\pi}$	$\frac{\Delta M}{\pi}$	$\frac{M}{\pi}$
At 7 cm	139.6579	.0151	$139.6730 \times 1 = 139.6730$
8	127.3947	.0144	$127.4091 \times 2 = 254.8182$
9	116.7852	.0136	$116.7988 \times 3 = 350.3964$
10	107.4885	.0129	$107.5014 \times 4 = 430.0056$
11	99.2607	.0122	$99.2729 \times 3 = 297.8187$
12	91.9204	.0116	$91.9320 \times 2 = 183.8640$
13	85.3291	.0111	$85.3402 \times 1 = 85.3402$
			$\Sigma(M \pi) = 1741.9161$
			$\Sigma(M \pi) \div 16 = 108.8698$
			$M_0 \pi = 107.4885$
			$\Delta M \div \pi = 1.3813$
Calculated above by (27), Example 19,			$\Delta M \div \pi = 1.3808$
		Difference	$= .0005$

This difference of five parts in a million of the value of M_{ab} is an extremely small discrepancy. The value calculated by the method of decomposing the coil into several coils of square section is very accurate if the several values of M are calculated with sufficient accuracy, and we may be sure of the value above to less than one part in a million, as the average value of the corrections ΔM for the separate square sections is less than a hundredth part of ΔM for the whole coils A, B . The result by formula (30) differs slightly from that by (27), being .0013 less.

This difference is mainly due to the sixth differentials, neglected in (30), which example (19) shows to be .0018. If we add this to the value obtained by (30) we have 1.3813, agreeing exactly with

Mutual Inductance of two Coils by Five Different Formulae at Three Different Distances.

a=25, b=4, c=1; d=6, 10 and 20 cm.

	d=6 cm.			d=10 cm.			d=20 cm.		
	Value of M	Error	Error parts in 100,000	108.8698	Error	Error parts in 100,000	53.5310	Error	Error parts in 100,000
Value of M ₀	158.0077			107.4885			53.1850		
	154.0718								
True Value of ΔM	3.9359			1.3813			.3460		
ΔM by formula (14), Rowland	3.6105	.3254	206	1.3429	.0384	35	.3435	.0025	5
ΔM " (19), Rayleigh	3.8344	.1015	64	1.3706	.0107	10	.3412	.0048	9
ΔM " (23), Lyle	3.8772	.0587	37	1.3755	.0058	5.5	.3458	.0002	0.4
ΔM " (30), Rosa-Weinstein	3.9139	.0220	14	1.3795	.0018	1.7	.3460	.0000	0
ΔM " (27), Rosa	3.9272	.0087	5.5	1.3808	.0005	0.5	.3425	.0035	7

The above results are shown graphically in Fig. 16.

the value above by the more laborious method of decomposition. This close agreement of (27) and (30), two expressions derived by different processes from two entirely different formulae, is very satisfactory.

Lyle's formula (22) for this case gives a value too small by .0058, Rayleigh's (19) too small by .0107, and Rowland's (14) too small by .0384. The accompanying table gives these results together with corresponding values for $d=6$ and $d=20$. Formula (27) begins to be in error before $d=20$, but for the smaller distances it is the most accurate of all. Formulae (27) and (30) supplement each other, and taken together they will give reliable results for all distances except when the coils are nearly in contact. When the coils are in contact or nearly so, formulae (31) and (32) are to be used.

COILS OF ONE LAYER. CURRENT SHEETS.

Formulae 26 and 30 apply when the depth is very small compared with the breadth of the coil, as in a single layer coil. In such a case the other formulae are scarcely applicable, as the error increases rapidly with a great divergence from square section. If we make $c=0$, as in a current sheet we can get an interesting check on the results of the formulae by using the formula for the self-inductance of current sheets, deriving from the values of several self-inductances the mutual inductance in question. Suppose we have two coaxial current sheets A, B each 5 cm long and with a radius of 25 cm, their centers being 10 cm apart. If a third current sheet C fill the gap between them it would be 5cm long. Let L_A be the self-inductance of A, which is also that of B or C, L_{AB} be the self-inductance of two sections AC together (or CB together) counting AC as two turns in series, and L_{ABC} be the self inductance of the three together in series. Also let M_{AC} be the mutual inductance of A on C and M_{AB} be the mutual inductance of A on B. We wish to find M_{AB} . Remembering that the self-inductance of a coil of several turns is equal to the sum of the several self-inductances plus the mutual inductances of the several turns on one another, we see that:

$$\begin{aligned}
 L_{ABC} &= 3L_A + 4M_{AC} + 2M_{AB} \\
 \text{Also } L_{AB} &= 2L_A + M_{AC} \\
 \therefore L_{ABC} &= 3L_A + 2(L_{AB} - 2L_A) + 2M_{AB} \\
 \text{or, } M_{AB} &= \frac{L_{ABC} + L_A - 2L_{AB}}{2}
 \end{aligned} \tag{33}$$

If we calculate the three self-inductances L_{ABC} , L_{AB} , and L_A , for the current sheets by an accurate formula we can obtain M_{AB} , and thus obtain a check on the values calculated for the same case by (27) and (30).

These self-inductances may be calculated with great precision by the formula of Coffin¹⁹ or of Lorentz.¹⁹

The results are as follows, taking the above dimensions, with 50 turns of thin tape 1 mm wide on each coil:

$$\begin{aligned} L_{ABC} &= 4,774,460\pi \\ L_A &= 798,457\pi \\ \text{Sum} &= 5,572,917\pi \\ 2L_{AB} &= 5,023,808\pi \\ \therefore 2M_{AB} &= 549,109\pi \\ \frac{M_{AB}}{\pi} &= 274,554.5 \text{ cm} \end{aligned}$$

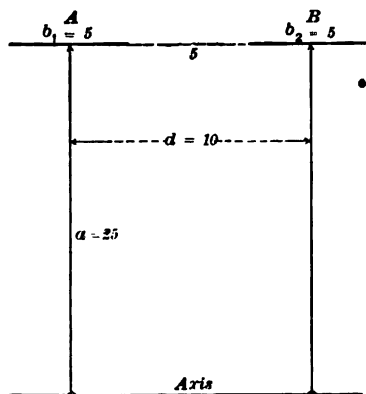


Fig. 17

Formula (26) for the case of $c=0$, extended to include two terms of the sixth and eighth degree (the fourth and fifth below) and a term depending on the differential coefficients of the eighth order (the tenth) is as follows:²⁰

$$\begin{aligned} \frac{\Delta M}{\pi n^2} &= 4a \left[\frac{b^2}{12a^3} + \frac{b^2}{32a^3} \left(\log \frac{8a}{d} - \frac{11}{6} \right) - \frac{15b^2 d^2}{1024a^4} \left(\log \frac{8a}{d} - \frac{97}{60} \right) \right. \\ &\quad + \frac{175b^2 a^2}{2(128)^3 a^4} \left(\log \frac{8a}{d} - \frac{54}{35} \right) - \frac{3675b^2 a^2}{(128)^3 a^4} \left(\log \frac{8a}{d} - \frac{3793}{2520} \right) \\ &\quad \left. + \frac{b^4}{960a^3 a^2} + \frac{b^4}{60a^3} - \frac{b^4}{1024a^4} \left(\log \frac{8a}{d} - \frac{187}{60} \right) + \frac{b^6}{168a^4} + \frac{b^6}{360a^4} \right] \quad (34) \end{aligned}$$

The following is the calculation of ΔM for the two single layer coils above, considered as current sheets:

$$a = 25, \quad b = 5, \quad d = 10 \qquad \log_e \frac{8a}{d} = 2.99573$$

¹⁹ Bulletin of Bureau of Standards, 2, p. 136; 1906.

²⁰ Rosa, this Bulletin, p. 351, equation (51).

$$\begin{aligned}
 (1) \frac{b^2}{12d^2} &= \frac{25}{1200} = .0208333 \\
 (2) \frac{b^2}{32a^2} \left(\log \frac{8a}{d} - \frac{11}{6} \right) &= \frac{25 \times 1.1624}{20000} = .0014530 \\
 (4) \frac{175b^2d^4}{2(128)^2a^6} \left(\log \frac{8a}{d} - \frac{54}{35} \right) &= \frac{175 \times 250000 \times 1.45}{2(128)^2 25^4} = .0000080 \\
 (6) \frac{b^4}{960a^2d^6} &= \frac{25^2}{960 \times 25^2 \times 100} = .0000104 \\
 (7) \frac{b^4}{60d^4} &= \frac{625}{60 \times 10,000} = .0010417 \\
 (8) \frac{-b^4}{1024a^4} \left(\log \frac{8a}{d} - \frac{187}{60} \right) &= + \frac{625 \times 0.121}{1024 \times 625^2} = .0000002 \\
 (9) \frac{b^8}{168d^8} &= \frac{1}{168.2^8} = .0000930 \\
 (10) \frac{b^8}{360d^8} &= \frac{1}{360 \times 2^8} = \frac{.0000108}{+.0234504} \\
 (3) -\frac{15b^2d^2}{1024a^4} \left(\log \frac{8a}{d} - \frac{97}{60} \right) &= \frac{37,500 \times 1.379}{1024 \times 25^4} = -.0001293 \\
 (5) -\frac{3675b^2d^4}{(128)^2a^6} \left(\log \frac{8a}{d} - \frac{3793}{2520} \right) &= \dots\dots\dots = \frac{-.0000004}{-.0001297} \\
 \text{Sum of 10 terms} &= +.0233207 \\
 4an^2 &= 250,000 \\
 \therefore \frac{\Delta M}{\pi} \dots &= 5,830.2 \text{ cm}
 \end{aligned}$$

By formula (1) we find M_0 , the mutual inductance of two circles at the centers of these current sheets, for which $a=25$, $d=10$, to be 268,721.3 cm. We thus have

$$M = M_0 + \Delta M = \pi(268,721.3 + 5,830.2) \text{ or } \frac{M}{\pi} = 274,551.5 \text{ cm}$$

This is less than the value found above from the self inductance formula by 3 cm, which is about one part in 100,000. This very

close agreement in the results of two entirely independent formulæ is very satisfactory.

We have also calculated ΔM for this case by formula (30) putting $c=0$. The result is as follows:

$$\frac{\Delta M}{\pi} = 4an^2 \times .023217 = 5,804.2$$

This is a little less than the value found by (34), as we should expect, as (30) does not take account of the sixth and eighth differentials, as does (34). We can, however, pick out these two terms from (34) and add them to (30) and see what difference remains in the results. From the above calculation we see that these two terms (the ninth and tenth of 34) amount to $.0001038 \times 4 an^2$. Adding to the result by (30) we have

$$\frac{\Delta M}{\pi} = 250,000 \times .0233208 = 5,830.2 \text{ cm}$$

which is exactly the value found by (34). The remaining discrepancy of one part in a hundred thousand represents the terms depending on differential coefficients of order higher than the eighth. If the coils were farther apart or narrower, these higher terms would be still smaller. When they are nearer or broader they are more important. For two current sheets 10 cm wide and 5 cm between there is a discrepancy of one part in 20,000 due to these terms of degree higher than the eighth. Such cases (where the coils are close together) should of course be computed by (33) rather than (34) for the highest accuracy, but (34) is more convenient and amply accurate for most purposes.

EFFECT OF INSULATION ON THE WIRE.

We have so far assumed in calculating ΔM that the current is uniformly distributed over the cross section of the coil. How much is ΔM altered if the current flows through round wires covered by insulation? We have found above that ΔM for a pair of coils, A, B, of section 4×1 cm and 10 cm between centers amounts to 1.38 where M is 108.87. That is, the mutual inductance is 1.3% greater when the current is distributed uniformly over the section than

when it is concentrated at the centers of section $O_1 O_2$. If, however, the current flows in four circles at the centers of the four small squares making up the section, as in $A_2 B_2$, the excess for uniform distribution is only .013, or about a hundredth of 1 per cent. If the current were to flow through round wires of considerable section, as indicated in $A_3 B_3$, the difference would be still smaller. Hence four heavily insulated wires, with an exterior diameter of 1 cm, make up a coil which is equivalent to a winding of square wires with infinitesimal insulation to within about one part in ten thousand in the value of M , for the dimensions of this example.

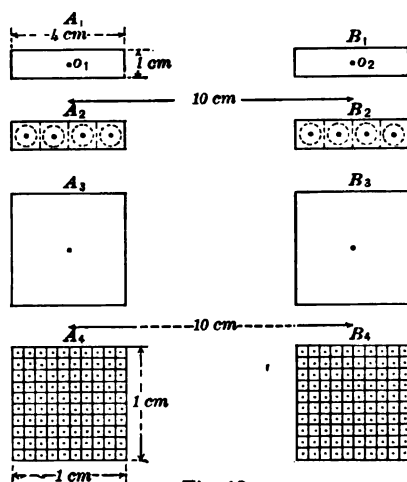


Fig. 18

If the cross section is square, 1 cm on a side, and a is 25 cm and d is 10 cm, ΔM is, as in the last case, .013, when M_0 is 108.87, or about one part in 10,000; that is, the difference in M when the current is concentrated at the center in one case and uniformly distributed over the square in the other is a little more than one in 10,000. If, however, the section be filled with insulated wire of one millimeter diameter, so that there are 100 small squares, ΔM for the whole coil will be proportional to the mean value of ΔM for the small squares. This will be less than 1 per cent of the former value, as the correction is proportional to b^3 and b^4 . Hence the difference in M between the case of uniform distribution of current

over the section and having it concentrated in 100 insulated wires of 100 mm diameter is certainly less than one part in a million of M , for the given case. We may be sure therefore that in every practical case we are justified in employing our formulæ for ΔM , even though the current is not actually distributed over the entire cross section of the wire.

In taking the dimensions of the section we must take the entire space filled by the wire, *including the insulation*, on the wires forming the outer layers. That is, if the width of the section is made so that a given number of wires fills it closely, we should take b as the total width of the channel; c will be half the difference between the diameter before and after winding.

In order to reduce the uncertainty of the position of the wire and increase the uniformity of the winding, enameled-covered wire, which has a very thin and uniform covering, should be used. The effect of a slight lack of uniformity in winding parallel to the axis is eliminated by interchanging the coils in measuring the mutual inductance. The effect of a lack of uniformity radially is eliminated by determining the equivalent radius experimentally by comparison with a single layer coil.

The effect of slight errors in the values of b and c will be least when the section is square. The section should of course be as small as is consistent with obtaining a suitable value of M and keeping the resistance within reasonable limits.

The close agreement of independent formulæ when applied to particular cases inspires confidence in the formulæ, and it is for this reason that we have carried out the calculations in the above examples further than would be strictly necessary for use in experimental work. We think that, by properly choosing the dimensions of the coils, and decomposing them in the calculations into two or more parts when the sections are relatively large, that there is no difficulty in calculating the mutual inductances of a pair of coils of equal radii to one part in 50,000 or 100,000. By the methods of Bosscha,²¹ Rayleigh,²² and Lyle²³ the mean radius can be accurately

²¹ Bosscha: Pogg. Annalen, **93**, p. 402; 1854.

²² Rayleigh, Sci. Papers, Vol. II, p. 184.

²³ Lyle, Phil. Mag., **3**, p. 310; 1902.

determined with reference to a standard which is capable of very precise direct measurement. The distance apart can also be determined with very great accuracy, so that the possibilities of accuracy in the measurements as well as in the experimental determinations of the mutual inductance justify the use of very accurate formulæ in the calculations. Thus in absolute determinations of resistance by the measurement of self-inductance, results may be obtained not only by single layer coils but also by means of coaxial coils of several layers.



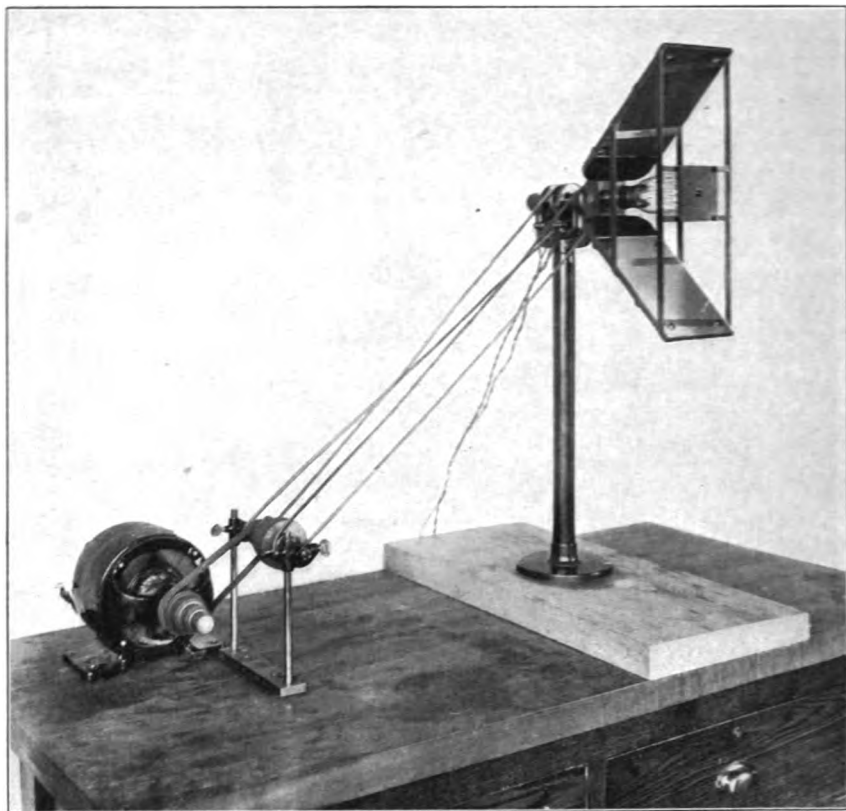


Fig. 1.—*Rotating Mirror System.*

ON THE DETERMINATION OF THE MEAN HORIZONTAL INTENSITY OF INCANDESCENT LAMPS BY THE ROTATING LAMP METHOD.

By Edward P. Hyde and F. E. Cady.

INTRODUCTION.

There are several methods in use at the present time for the determination of the so-called mean horizontal intensity of incandescent lamps, i. e. the mean intensity in the plane perpendicular to the axis of the lamp and passing through its center. The oldest of these methods, and one which is equally applicable to other light sources, consists in measuring the intensity at equal angular intervals in the horizontal plane, and either taking the mean of the observed values, or plotting the observations and determining the mean radius vector of the curve drawn through the plotted points. A second method particularly adapted to the incandescent lamp because of the constancy of its curve of mean horizontal intensity was developed from the first method as an abbreviation of it. According to this second method photometric measurements are made in a single fixed direction, and the mean horizontal intensity is computed from mean horizontal reduction factors previously determined for the different types of lamps. Dyke has recently shown,¹ however, that even for lamps of a single type of filament individual differences of such magnitude occur as to produce serious errors in the computed values of mean horizontal intensity. This fact in conjunction with the inconvenience of the necessary orientation of the lamps on the photometer bar in making the measure-

¹ Phil. Mag., 9, p. 136; 1905.

ments has rendered the method of little practical value, and consequently its use at present is quite restricted.

In a modified form of this method which is used in Germany, the direct light in a definite direction from the lamp is supplemented by light reflected from two mirrors placed behind the lamp and inclined at an angle of 120° , so that an illumination is produced on the photometer screen, which is approximately proportional to the mean of the intensities in the three directions. Liebenthal has shown² the possible errors incident to this method, for the types of lamps studied, to be as large as 5 or 6 per cent when the lamps are definitely oriented, and 13 per cent when the lamps are set at random.

A third method, and one which is in almost universal use in this country in all practical determinations of the mean horizontal intensity of incandescent lamps, consists in spinning the lamp about its axis at a uniform speed of 180 revolutions per minute. The suggestion of spinning the lamp is said to have been made first by Prof. C. R. Cross at the National Electrical Congress held in Philadelphia, in 1884, but the writers have been unable so far to find a record of such a suggestion. Thirteen years later the Committee on Standardization of the American Institute of Electrical Engineers recommended³ this method, suggesting a speed of 120 revolutions per minute. In recent years a speed of 180 revolutions per minute has been used quite universally, but an attempt to trace the authority for this modification has not as yet met with success.

A fourth method devised by Liebenthal⁴ and used at the Physikalisch-Technische Reichsanstalt consists in rotating a pair of mirrors about the lamp held in a stationary position. The lamp is mounted with its axis horizontal and coincident with the photometric axis, the tip of the lamp being turned toward the photometer screen. The two mirrors, inclined at approximately 90° to each other, reflect to the photometer screen light emitted from the lamp in a direction perpendicular to its axis. The direct rays from the tip of the lamp are prevented from reaching the photometer by a small screen. The mirrors rotate about the axis of the lamp and reflect to the screen a quantity of light proportional to the mean

² *Zs. für Instrumentenkunde*, 19, p. 193; 1899.

³ *Trans. Amer. Inst. of Elec. Eng.*, 14, p. 90; 1897.

⁴ *Loc. cit.*

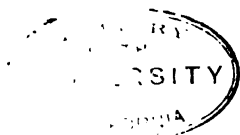
horizontal intensity of the source. This method necessitates, of course, the determination of the reflection coefficients of the mirrors, which is accomplished by means of a standard incandescent lamp, the mean horizontal intensity of which is known from a previous determination by the first method mentioned above.

Of the above four methods by far the most convenient in commercial testing is that of rotating the lamp, because but one measurement is required, and there is no question about the orientation of the lamp. Moreover, by simple mechanical devices a reversal of the direction of rotation may be obtained so that lamps fitted with screw bases can automatically be screwed into and out of the socket, thus materially lessening the time required for measurement. Although the method of rotating the lamp is the most convenient it has been open to serious objection on account of possible errors which have never been investigated, so that while the method is in quite general use in this country there are many who are in doubt in regard to its accuracy, and its adoption abroad has been restricted.

The two possible sources of error of this method are (1) the possible distortion of the filament, and (2) the effect of the flicker, which is perceptible in nearly all types of lamps when rotating at 180 revolutions per minute, and which is extremely bad in those types in which the horizontal distribution curve deviates greatly from a circle. Besides the possible error produced by the flicker, the accuracy of a setting is diminished considerably and the eye is rapidly fatigued.

DESCRIPTION OF APPARATUS.

In order to determine definitely the errors due to distortion of the filament and to flicker, an investigation has recently been carried out in the Photometric Laboratory of the Bureau of Standards. A pair of rotating mirrors, Fig. 1 (frontispiece), somewhat similar to those described above as in use at the Reichsanstalt, but with one essential difference, were constructed in the instrument shop of the Bureau. Instead of mounting the lamp in a stationary position between the mirrors, the apparatus was so designed that the lamp could be rotated about its axis quite independently of the rotation of the mirrors. This arrangement obviated the necessity of a separate determination of the coefficient of reflection of the mirrors, which is



liable to error unless the mean horizontal intensity of the standard lamp employed is obtained by the point to point method from a very large number of readings.

By designing the apparatus in such a manner that the mirrors and lamp could be rotated independently a substitution method could be employed and the two elements, bending and flicker, could be separately determined irrespective of the reflection coefficient of the mirrors. The assumption is made, of course, that the coefficient of reflection remains the same independent of the speed of rotation. This assumption was subsequently verified by direct experiment. A lamp with an approximately uniform horizontal distribution curve was placed in a stationary position in the socket and the mirrors were rotated first at the lowest possible speed to eliminate flicker, which was about 300 revolutions per minute, and then at 900 revolutions per minute. The mean photometric settings under the two conditions agreed to within 0.1 per cent, which was less than the possible error of observation.

Since the constancy of the reflection coefficient was of such importance in the experiments, great care was exercised in the design of the mirror system to insure rigidity. Fig. 2 is a vertical section of the mirror system through the axis of rotation. The metal plates, *A*, on which the mirrors were mounted, were quite heavy, tapering from a thickness of 6 mm at the outside end to a thickness of 10 mm at the junction with the hub. These plates moreover, were reenforced by longitudinal and transverse webs which added greatly to the rigidity of the mirror supports. Over the surface of each plate a thin layer of plaster of Paris was spread in which the mirror, *M*, was allowed to settle. This formed a hard surface in contact with the mirror at every point and thus prevented any distortion of the mirror under the action of centrifugal forces. Four small bolts, *B*, through each mirror, prevented a possible slipping of the mirrors from the plaster cast.

The frames of the two mirrors were joined at the top and bottom by stay rods, *C*, from two of which a small disk, *D*, was suspended to prevent the direct light in the direction of the tip of the lamp from reaching the photometer. The angle between the two mirrors was such that for a definite distance from the photometer the rays of light emanating from the center of the lamp and normal to the

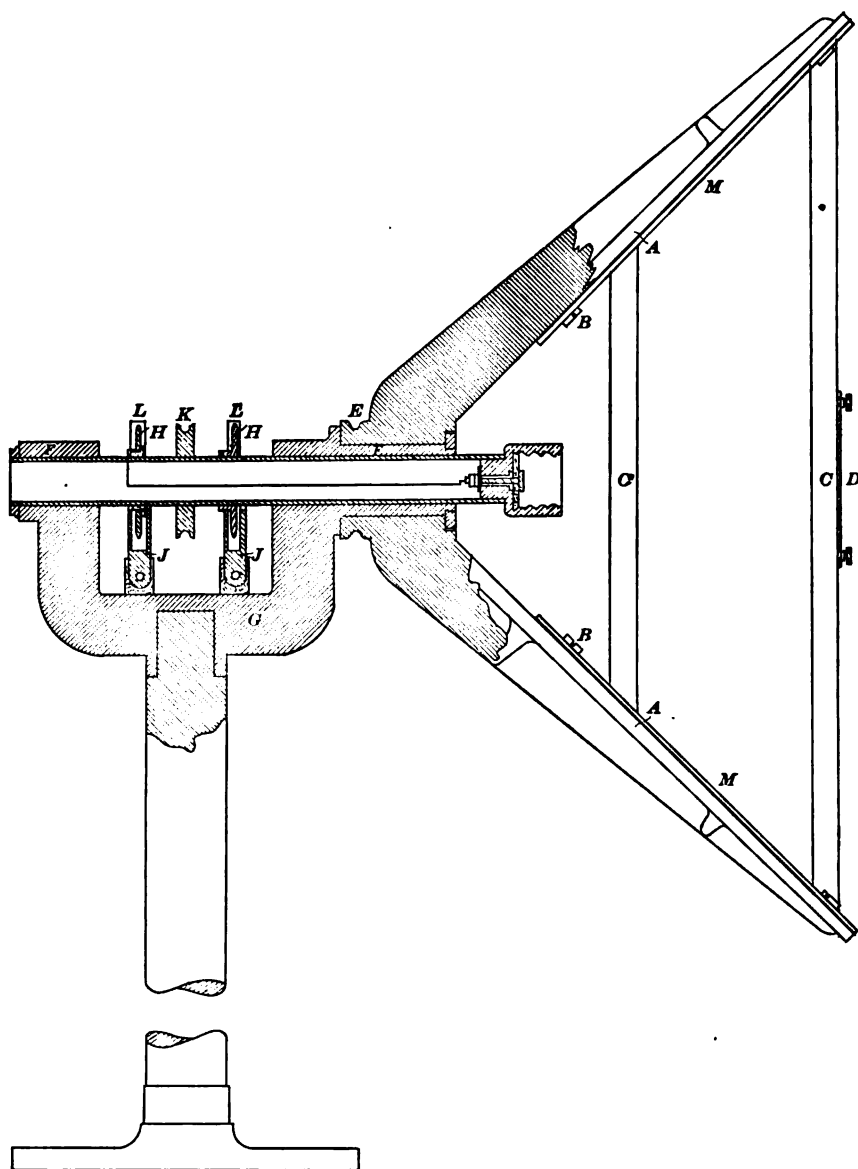


Fig. 2

Vertical Section of Rotating Mirrors through Axis of Rotation.

axis of the lamp were reflected at the mirrors and were incident on the photometer screen at its center.

The entire mirror system was driven by a round belt operating in a grooved pulley, *E*, on the hub. The bearing surface on which the hub rotated was turned down from a boss, *F*, on one arm of a U-shaped casting, *G*. By drilling through the bosses, *FF*, on the two arms of the casting a bearing surface for the shaft carrying the lamp socket was formed. The terminals of the socket were connected to amalgamated copper disks, *H*, mounted on the shaft, and these disks rotated in mercury cups, *J*, supported on the casting. The lamp was driven by means of a grooved pulley, *K*, mounted between the copper disks. This design allowed quite independent rotation of the mirrors and the lamp about the same axis, the mercury contacts preventing any variable resistance when the lamp was

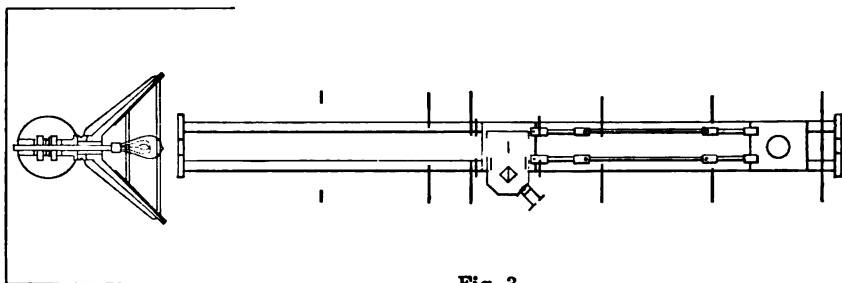


Fig. 3.

Diagrammatic Sketch of Rotating Mirror System in Position.

rotating. Shields, *L*, were constructed over the mercury cups to prevent excessive throwing of mercury at high speeds of rotation of the lamp. The U-shaped casting carrying the mirror system and the lamp was supported on a steel shaft which was sweated into a large base. The base was screwed down to a heavy oak board, which was clamped on a table at the end of a photometer bench in such a position that the axis of rotation of the mirrors and lamp coincided very closely with the photometric axis. Fig. 3 is a diagrammatic sketch showing the rotating mirror system in position.

Since with the introduction of mirrors the system of diaphragms commonly used on the photometer was inadequate, a house of black velvet was built around the rotating mirrors so that no light could reach the photometer except that directly reflected from the mirrors and the stray light reflected from the black velvet.

The electrical measurements were made by the aid of a potentiometer. In the photometric measurements the substitution method was used, the comparison lamp remaining at a fixed distance from the photometer and moving with it as a single system.

GENERAL METHOD.

In order to determine with the above apparatus the change in mean horizontal intensity due to bending and flicker, both separately and together, the following series of measurements was made on lamps of a number of different types: (*a*) mirrors rotating at a sufficiently high speed to eliminate flicker (in some cases over 800 revolutions per minute), lamp at rest; (*b*) mirrors rotating at 180 revolutions per minute, lamp at rest; (*c*) mirrors at rest, lamp rotating at 180 revolutions per minute; (*d*) mirrors at rest, lamp rotating at 500 or 600 revolutions per minute; then the entire cycle in the reverse order. In all the measurements in order that the observer at the photometer should not be influenced by his readings a second observer noted and recorded the settings on the bar, and the observer at the photometer was not informed of the results until the entire series of measurements had been completed.

If there were no error due to bending, and no error due to flicker, all four of the above sets of readings would be the same. A comparison of (*a*) and (*b*) shows the effect of flicker separated from that of distortion of the filament; a comparison of (*a*) and (*c*) shows the combined error due to flicker and bending at 180 revolutions per minute. This combined error is that which occurs in practice when measurements are made on lamps rotating at 180 revolutions per minute. A comparison of (*b*) and (*c*) shows the effect of bending at 180 revolutions per minute; and a comparison of (*a*) and (*d*) shows the effect of bending at 500 or 600 revolutions per minute.

In all, ten different types of incandescent lamps were studied, as follows:

- A- 50 volt, 16 cp, horseshoe filament lamp.
- B-220 " 16 " double oval anchored filament lamp.
- C-110 " 16 " oval anchored filament lamp.
- D-110 " 16 " double carbon filament lamp.
- E-110 " 16 " double flattened coil filament lamp.
- F-110 " 16 " double round coil filament lamp.

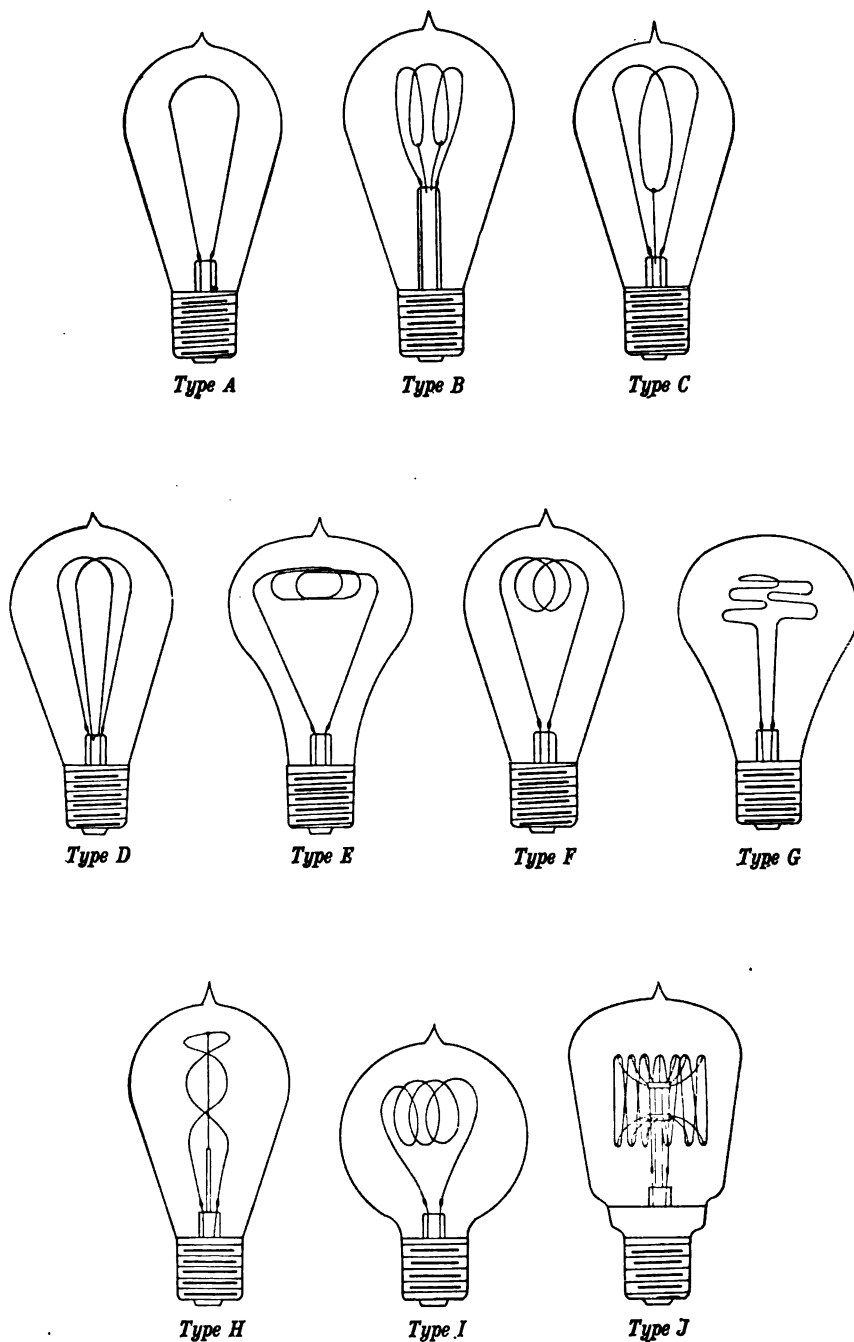


Fig. 4.

Different Types of Lamps Investigated.

G-110 volt, 16 cp, downward light filament lamp.

H-110 " 16 " spiral filament lamp.

I-118 " 16 " round bulb, helical filament lamp (concentrated carbon deposit).

J-110 " 22 " tantalum lamp.

The forms of the filaments of these ten different types of lamps are shown in Fig. 4.

The results of the investigation can be summarized under the two heads—"Effect of Bending," and "Effect of Flicker."

EFFECT OF BENDING.

At speeds of 500 or 600 revolutions per minute very appreciable changes in the mean horizontal intensity due to bending were observed, in some cases the effect being to decrease the mean intensity, in others to increase it. The greatest difference due to bending was observed in a double carbon filament lamp which had accidentally been rotated for a few seconds at about 800 revolutions per minute and was afterward brought to a speed of about 600 revolutions per minute. This lamp showed a decrease in mean horizontal intensity of about 2 per cent, but after the lamp was stopped rotating it was noticed that one loop of the filament had touched the bulb and had sealed itself there, remaining permanently in that position. Except for this one lamp the results on the effect of bending at a speed of 550 revolutions per minute for the lamps of the different types studied are summarized in Table I. In the first column the type of lamp is given. The second column contains the individual determinations of the effect, (+) indicating that the effect of rotation was to increase the mean horizontal intensity, (—) that the effect was to decrease it. In the third column are given the means of the individual determinations for each type of lamp.

It is seen from Table I that the greatest change in mean horizontal intensity due to a bending of the filament occurred in the double carbon type of filament in which on the average a decrease of 0.9 per cent, was observed. The greatest increase was 0.8 per cent, which was the mean value found for the downward light type of filament. The oval anchored filament showed a decrease of about 0.5 per cent. At a speed of 180 revolutions per minute the change in mean horizontal intensity due to bending was so small for all

types of lamps studied as to lie within the range of experimental error, which was probably several tenths of one per cent.

TABLE I.

Effect of Bending.

Type of lamp	Change in mean horizontal intensity	Means	Type of lamp	Change in mean horizontal intensity	Means
A	$\pm 0.0\%$	$\pm 0.0\%$	E	$+0.3\%$	
B	-0.1	-0.1		+0.6	
C	-0.6			+0.4	+0.2%
	-0.3			± 0.0	
	-0.4			± 0.0	
	-0.3	-0.5		± 0.0	
	-0.7		F	-0.2	
	-0.4			+0.1	± 0.0
	-0.6			± 0.0	
D	-0.7		G	+0.9	+0.8
	-1.1	-0.9		+0.6	
	-0.8		H	-0.1	
				+0.1	-0.1
				-0.3	

The change in mean horizontal intensity produced by bending is to be explained by the difference in the mean effective length of the filament in the two cases. A straight vertical filament would have a maximum intensity in the horizontal plane. If the filament were bent due to spinning, its effective length would be decreased; or, looking at the question in another way, the light in the horizontal plane would be emitted at some angle, θ , to the normal, and consequently the intensity for each element of filament would be decreased in the ratio of $\cos \theta:1$. For a straight vertical filament we could be certain that the effect of bending would be to decrease the mean horizontal intensity. In the case of complex filaments, if the filament simply bent as a whole, we could be certain that the bending in all types of lamps would increase or decrease the mean horizontal intensity depending upon whether the mean horizontal intensity is less or

greater than the mean latitudinal intensity on either side of the horizontal. Since, however, superimposed upon the effect of bending of the filament as a whole is the effect due to the relative bending of the parts of the filament, it is difficult to state *à priori* the direction or magnitude of the total effect. It is significant, however, that the two types of lamps in which the greatest increase and greatest decrease in mean horizontal intensity were observed are those in which the mean horizontal intensity is notably a minimum, and a maximum, respectively.

The tantalum lamp showed a queer effect which it is interesting to note in passing. Although one might expect a decrease in mean horizontal intensity on rotation, no such effect was found, the observations indicating a slight increase. The magnitude of the effect was small, but since in every one of a number of determinations the sign of the effect was the same it is probable that there is a real increase in mean horizontal intensity. The peculiar feature to which attention should be called, however, is the apparent increase in resistance of tantalum lamps on rotating them. At least a half dozen lamps were tested, each of them several times, and without exception on rotating the lamps the current decreased, although the voltage was maintained constant. The amount of change was not the same for all lamps, nor indeed the same at different times for any one lamp, but ranged from 1 per cent to only a few tenths of 1 per cent.

EFFECT OF FLICKER.

According to Talbot's law, if the retina is excited with intermittent light recurring periodically, and if the period is sufficiently short, a continuous impression will result which is the same as that which would result if the total light received during each period were uniformly distributed throughout the whole period. Assuming this law, which has recently been investigated⁵ by one of the present writers and shown to be true for the extreme case of a rotating sector disk, the method of rotating the lamp would give the true mean horizontal intensity if the speed of rotation were sufficiently high to eliminate all flicker, provided there is no distortion of the filament. Matthews, by direct experiment,⁶ showed Talbot's

⁵ Bulletin of Bureau of Standards, 2, p. 1; 1906.

⁶ Physical Review, 6, p. 55; 1898.

law to hold under these conditions. But in almost all practical cases it is impossible to rotate the lamp at a sufficiently high speed to eliminate flicker without introducing errors due to distortion of the filament. Since we are not justified in extrapolating from Talbot's law to include cases in which there is a perceptible flicker, it is necessary to test by experiment whether or not the eye integrates a flickering light.

The most surprising results of the investigation were found in studying the effect of flicker on the apparent mean horizontal intensity of the different types of lamps. The common oval anchored type of filament shows comparatively little flicker when rotating at 180 revolutions per minute, and consequently, when small differences between sets (*a*) and (*b*) were found with this type of lamp they were ascribed to observational errors, since the difficulty of setting with a flickering light is much greater than when the fields of view in the photometer appear steady. When, however, with other types of lamps differences amounting to 4 per cent occurred it became evident that the effect was not to be ascribed to the probable error of observation, for every one of 5 or 10 readings with the mirrors at high speed would lie entirely without the limits of the readings at 180 revolutions per minute. Moreover, with different observers the effect of the flicker was found to be quite different. Thus, while one of us read the flickering light about 4 per cent too high, the other read about 3 per cent too low, a difference of 7 per cent. Moreover, each observer persisted in his habit, so that when, about this time, a number of lamps submitted for test were measured it was subsequently found on looking over the results that while the values obtained by the two observers agreed excellently for most lamps, there were large differences, always in the same direction, for lamps in which there was a disagreeable flicker.

Other observers were tried, of whom some were experienced in photometry, and all were experienced in other lines of physical investigation, but in almost every case there was a decided preference for either higher or lower values with the flickering light. All of the measurements referred to above were made with a Lummer-Brodhun contrast photometer, and in every case there was a color match on the two sides of the screen. Subsequently a number of measurements were made by three observers with a very good Leeson disk

furnished the Bureau by the Electrical Testing Laboratories of New York. Strangely enough, with this photometer each of the three observers changed the sign of his error as compared with his measurements on the Lummer-Brodhun photometer. Of course, the error of observation with a badly flickering light, such as is obtained with a double flattened or a downward light filament, is quite large, but there is no doubt in our minds that the error due to flicker is a very definite one, characteristic of the individual. It would not seem, however, that the effect is a true physiological effect in the same sense as Talbot's law for nonflickering illuminations. It is not that different eyes integrate a flickering light differently, but rather that they do not integrate at all. The explanation would seem to be this: Due to the inability of the eye to integrate a flickering light, an infinite number of different independent intensities are perceived. On any one of these a setting could be made, but having chosen some one point in the series of fluctuating intensities the criterion persists, and the more observations the individual makes the more probable it is he will continue to set in the same way. One of the observers previously mentioned was conscious of two different criteria by either of which he was able to make consistent settings. According to one criterion he set too low; according to the other, too high.

Apart from the explanation of the effect of flicker the important fact which the investigation makes clear is, that in the case of lamps in which the horizontal distribution curve deviates considerably from a circle so that a bad flicker results when the lamp is rotated at 180 revolutions per minute, some other method for determining the mean horizontal intensity must be employed. Otherwise, large errors amounting to many per cent may arise.

It is impossible to state what the errors may be because they depend entirely upon the individual. It may be that some observers may obtain erroneous settings with such a small flicker as that incident to an oval anchored filament lamp when rotating at 180 revolutions per minute, but it is very improbable that any large errors would be made except with a badly flickering light, such as occurs with the double flattened or downward light lamps at 180 revolutions per minute. It is true that these two types of lamps in which the flicker is most pronounced are not rated at mean horizontal intensity

by their manufacturers, but since in accordance with many specifications all lamps are purchased on this one basis, it becomes necessary to determine the mean horizontal candlepower of these lamps. Moreover in some other types of lamps which are usually rated on a basis of mean horizontal intensity, such as round bulb lamps with helical filaments, bad flickers occur and consequently large errors may result in the determination of the mean horizontal intensity by rotating the lamp at 180 revolutions per minute. The great danger in making measurements with a flickering light is that the observer has no easy way of determining the magnitude of the error which he may be making, and consequently there is always an uncertainty in the results.

Since none of the filaments of the different types of lamps investigated suffered any apparent injury at the high speed of 500 revolutions per minute and the effect of bending was in no case more than 1 per cent, it was thought desirable to investigate possible errors incident to the determination of mean horizontal intensity by rotating the lamps at a speed double that in use at present, i. e. at 360 revolutions per minute. Before doing this, however, some experiments were made on the mechanical effects of such a speed on lamps the filaments of which had drooped, due to prolonged burning in a horizontal position. Three 50 cp, 110 volt lamps, each of which had burned several hundred hours and had fallen in intensity below 80 per cent of its initial value, showed excessive drooping. In one lamp the filament was not more than 3 or 4 mm from the bulb; in each of the other two the filament was about 5 or 6 mm from the bulb. These lamps, which had long heavy filaments and which had given evidence of their susceptibility to bending under the influence of mechanical forces by their excessive drooping under the action of gravity, would seem to represent the most unfavorable case for a high speed rotation. The lamp in which the filament was within 4 mm of the bulb stood a speed of 300 revolutions per minute. At about 350 or 400 revolutions per minute the filament touched the bulb and melted it at the point of contact, allowing air to enter, which caused the lamp to burn out. The other two lamps were rotated at increasing speeds up to about 650 or 700 revolutions per minute before the filament touched the bulb, in every case the result of the contact being the same. Since these lamps would probably

have been rejected on most specifications before dropping to 80 per cent because of the excessive drooping, it would seem that a speed of 360 revolutions per minute would be quite safe in all practical cases. Although measurements made on different types of lamps indicated no determinate permanent change in the horizontal intensity, the effect of continued burning, while rotating at a speed of 360 revolutions per minute, was not investigated.

After showing that a speed of 360 revolutions per minute would probably produce no serious mechanical difficulties, samples of those types of lamps in which the effect of flicker is most pronounced and of those in which the bending is greatest, were measured at that speed in order to see whether the errors due to flicker could be obviated without introducing prohibitive errors due to bending. It was found that in several types of lamps the flicker was still so pronounced that differences of several per cent were obtained. It was therefore concluded that either a higher speed must be used or some other method employed.

Since it would not seem feasible to increase the speed much above 360 revolutions per minute an attempt was made to devise some other method by which an increased accuracy could be obtained. The following considerations suggest a method which was put into practice and found to be quite satisfactory: The error due to the flicker is a function not only of the speed, but of the difference between the maxima and the minima of the flickering light. No matter how great this difference the flicker will cease at sufficiently high speeds, but if the difference is small the error due to the flicker will be relatively small at all speeds and will entirely disappear when the speed has been increased even to a moderate value. In those types of lamps in which the error due to flicker was found to be large the differences between the maxima and the minima are great, amounting in one type to 70 per cent of the mean value, so that if in some way the extreme fluctuations in the intensity of illumination on the photometer screen could be lessened the resultant error would be diminished.

Except in the case of anomalous lamps extreme differences in intensity in different directions are not due to bad centering of the filament or to any effect connected with a displacement of the effect-

ive center of radiation from the axis of rotation; fluctuations due to these effects would in general be relatively small. The irregularity of the horizontal distribution curve is due primarily to the shape of the filament, and consequently, except for sharp reflections from the bulb or sharp shadows produced by the obscuration of parts of the filament, the maxima lie 180° apart, and in general the minima lie approximately at right angles to the maxima. Hence, if to the illumination produced by the direct light from the lamp on the photometer screen, an illumination produced by light emitted in a direction normal to the photometric axis be added, the resultant illumination will show a relatively small variation on turning the lamp. If the lamp is rotated at the customary rate of 180 revolutions per minute the flicker will be much less disagreeable and productive of error than that incident to the direct light alone when the lamp is rotated at double this speed.

In order to accomplish the above result in practice a single stationary mirror (Fig. 5) was placed in such a position that light

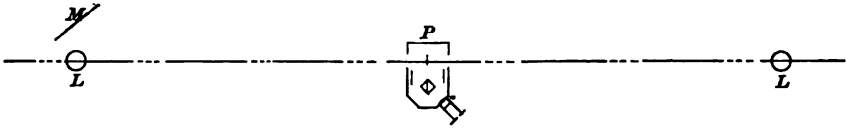


Fig. 5.

Sketch Showing Stationary Auxiliary Mirror in Position.

emitted at right angles to the photometric axis was reflected by the mirror and was incident on the screen in conjunction with the direct light from the lamp. Measurements made on the downward light lamp at 180 revolutions per minute by the two observers who had previously differed by about 7 per cent at this speed, showed an agreement well within the range of error of observation, which was probably less than one-half of 1 per cent. In a similar way the error due to flicker in all types of lamps in which the maxima and minima lie approximately at right angles to each other would become quite small even at a speed of 180 revolutions per minute. The results of the investigation at the Bureau show that a single mirror is sufficient for the types of lamps studied, but it is possible that other arrangements of mirrors might be devised that would also eliminate flicker.

The principal point to be emphasized is the general idea of combining the rotating lamp method with that of stationary mirrors.

In the subsequent paragraphs, however, the theoretical discussion will be confined to the case of a single mirror.

One or two round bulb lamps which had burned to 80 per cent in life test, and which were unique in having most of the carbon deposit concentrated on a small part of the bulb, showed bad flickers even with the addition of the mirror. This was to be expected, as measurements showed that, due to the absorption by the concentrated carbon deposit, there was a difference of 40 per cent between the intensity in the direction of the deposit and in a direction 180° from the deposit. In one direction the intensity was a maximum, in the other a minimum. With such anomalous lamps a very high speed must be used or some other method employed if accurate results are desired.

Having shown that the use of a single mirror diminishes the flicker to such an extent for all types of lamps investigated that the error due to flicker is negligible in commercial photometry, it only remains to show that the application of the mirror is entirely practical, taking into consideration the difference in path between the direct ray from the lamp and the ray by way of the mirror. Assuming that the two lamps are at fixed positions, and that the substitution method of measurement is employed, calculations were first made to determine what the error would be if the simple law of inverse squares were used and the distance to the effective center of radiation assumed to be intermediate between the distance of the lamp and the distance of its image from the screen. The candlepower values at definite distances on either side of the center (16-cp position) were computed on this basis and compared with the true candlepower values for the corresponding points, assuming the reflection coefficient of the mirror to be 0.85, and assuming the center of the mirror 15 cm from the axis of the lamp, measured in a direction perpendicular to the photometric axis. The results showed that for a bar 250 cm long the errors which would occur in measuring lamps ranging from about 4 cp to 150 cp would in no case be much greater than 1 per cent, but that with an increasing range of candlepower, particularly in the direction of lower candlepower values, the errors would rapidly increase. Since it is quite simple, however, to compute a rigorous candlepower scale which will be true for any position, assuming that the reflection coefficient of the

mirror is a known constant, no further investigation of the preceding method was made, such as the effect of a change in the reflection coefficient of the mirror. In the following paragraphs the rigorous method will be investigated in detail.

Let O , O' (Fig. 6) be the centers of the two lamps at a fixed distance, $2a$, apart, and let O'' be the image of O in the mirror BCD . The center, C , of this mirror is at a distance d from the lamp O , and lies on the line through O perpendicular to the photometric axis OO' . The mirror is turned in such a direction that the ray OC is incident on the photometer screen at its center when the photometer screen is at A , midway between the two lamps. For any other

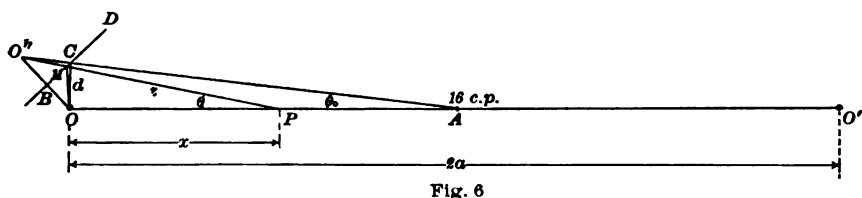


Fig. 6

position, P , of the photometer screen the path of the reflected ray will be OMP .

If we let J and J' be the intensities of the two lamps at O and O' , r the reflection coefficient of the mirror, x the distance from the lamp O to the photometer screen, z the distance from the image O'' to the photometer screen, and θ the angle of incidence of the reflected beam of light on the photometer screen, the equation of photometric balance at any position, P , will be

$$\frac{J}{x^2} + \frac{Jr \cos \theta}{z^2} = \frac{J'}{(2a-x)^2} \quad (1)$$

z^2 can be expressed in terms of x as follows:

$$z^2 = \overline{OO''}^2 + x^2 - 2 \overline{OO''}x \cos \angle POO'' \quad (2)$$

and if θ_0 is the value of θ corresponding to the middle point, A , of the photometer bar,

$$\overline{OO''} = 2\overline{OB} = 2\overline{OC} \cos \angle COB = 2d \cos \frac{1}{2}(90^\circ - \theta_0) \quad (3)$$

where θ_0 is determined by the equation

$$(4) \quad \tan \theta_0 = \frac{d}{a}$$

Moreover

$$(5) \quad \begin{aligned} \text{angle } POO'' &= 90^\circ + \text{angle } COO'' \\ &= 90^\circ + \frac{1}{2} (90^\circ - \theta_0) \end{aligned}$$

$$(6) \quad = 135^\circ - \frac{1}{2} \theta_0$$

Hence, all the terms in the right hand member of equation (2) can be expressed in terms of the variable, x , and the constants, a and d . $\cos \theta$ is determined from the sine relation,

$$(7) \quad \sin \theta = \frac{\overline{OO''}}{z} \sin POO''$$

in which the right hand member is a function of x , a , and d (equations (2), (3), (4), and (6)).

Having chosen the fixed distance, $2a$, between the two lamps, and the fixed distance, d , from the lamp O to the center of the mirror, $\overline{OO''}$ and the angle POO'' are determined and remain constant in the calculation of the candlepower values corresponding to different distances, x . From equation (1), which may be put in the form

$$(8) \quad \frac{J}{J'} = \frac{1}{(2a-x)^2} \frac{x^2 z^2}{r x^2 \cos \theta + z^2}$$

the ratio of J to J' may be calculated for different distances, x , and by choosing a value of J' which will bring a balance at the middle point of the bar ($x=a$) for $J=16$ cp, the candlepower values of J corresponding to other values of x may be readily computed, and the candlepower scale plotted.

In order to show how this equation works out in practice let us apply it to an actual case. On the photometer bench employed at the Bureau for commercial testing, the two lamps O and O' are 250 cm apart. Since the diameter of the largest bulb lamps that are ordinarily met with in the photometry of incandescent lamps does not

exceed 13 cm, the distance from the lamp to the center of the mirror may be taken as $d = 15$ cm, in which case the image of the lamp in the mirror will in no case be obscured by the lamp itself, provided the photometer screen is not brought below the position corresponding to 2 cp.

If we put $2a = 250$ cm, and $d = 15$ cm, we obtain, on substituting in equation (4);

$$\begin{aligned}\tan \theta_0 &= \frac{d}{a} = \frac{15}{125} \\ &= 6^\circ 50' 34''\end{aligned}\quad (9)$$

From equation (3)

$$\begin{aligned}\overline{OO''} &= 2d \cos \frac{1}{2}(90^\circ - \theta_0) \\ &= 2 \times 15 \cos \frac{1}{2}(90^\circ - 6^\circ 50' 34'') \\ &= 22.44 \text{ cm}\end{aligned}\quad (10)$$

From equation (5)

$$\begin{aligned}\text{angle } POO'' &= 135^\circ - \frac{1}{2}\theta_0 \\ &= 135^\circ - \frac{1}{2}(6^\circ 50' 34'') \\ &= 131^\circ 34' 43''\end{aligned}$$

Substituting these values in equation (2) we get

$$\begin{aligned}z^2 &= \overline{OO''}^2 + x^2 - 2\overline{OO''} x \cos POO'' \\ &= 503.6 + x^2 + 29.78x \\ &= (x + 14.89)^2 + 281.9\end{aligned}\quad (12)$$

Therefore equation (8) becomes

$$\frac{J}{J'} = \frac{1}{(250 - x)^2} \frac{x^2 ((x + 14.89)^2 + 281.9)}{rx^2 \cos \theta + (x + 14.89)^2 + 281.9} \quad (13)$$

where $\cos \theta$ is obtained from the relation (equation (7))

$$\begin{aligned}\sin \theta &= \frac{\overline{OO''}}{z} \sin POO'' \\ &= \frac{22.44}{\sqrt{(x + 14.89)^2 + 281.9}} \sin 131^\circ 34' 43''\end{aligned}\quad (14)$$

By assigning different values to x in equations (13) and (14) the ratio $\frac{J}{J'}$ may be computed for any reflection coefficient, r . Let us make $r=0.85$, which represents approximately the average value for good silver-back mirrors, and let us compute the ratio $\frac{J}{J'}$ for the distances, $x=65, 95, 125, 155$, and 185 cm. The results of the computation are given in the sixth column of Table II. If now a 16 cp standard is to balance against the comparison lamp at the middle point of the bar, $x=125$ cm, the required intensity J' of the comparison lamp is immediately obtained by putting $J=16$ for $x=125$ cm and computing the value of J' from the equation

$$\frac{J}{J'} = 0.6009 \quad (15)$$

from which $J' = 26.63$ cp.

By giving J' this value in the ratio $\frac{J}{J'}$ at other distances, the candlepower values corresponding to the respective distances $x=65, 95$, etc., are obtained. These are given in the seventh column of Table II.

TABLE II.

Values of $\frac{J}{J'}$ and J for Different Reflection Coefficients, r , and Different Distances, x .

x	$r=0.75$		$r=0.80$		$r=0.85$		$r=0.90$		$r=0.95$	
	$\frac{J}{J'}$	J	$\frac{J}{J'}$	J	$\frac{J}{J'}$	J	$\frac{J}{J'}$	J	$\frac{J}{J'}$	J
65	0.08425	2.138	0.08250	2.145	0.08082	2.152	0.07922	2.159	0.07767	2.165
95	0.2437	6.185	0.2381	6.192	0.2328	6.199	0.2272	6.206	0.2228	6.212
125	0.6305	16.00	0.6153	16.00	0.6009	16.00	0.5870	16.00	0.5739	16.00
155	1.6480	41.82	1.6072	41.79	1.5684	41.76	1.5313	41.74	1.4960	41.71
185	4.952	125.68	4.827	125.53	4.708	125.38	4.595	125.24	4.487	125.09

Of course, in plotting a candlepower scale the value of J for a great many more distances, x , must be used, but the values given in Table II indicate the method to be employed, and cover the range of candlepower likely to be used.

It is interesting now to determine what change in the candle-power scale is necessary corresponding to a different reflection coefficient. This determination is important for two reasons: (1) it indicates the degree of accuracy required in the initial measurement of the reflection coefficient of the mirror to be used, and (2) it indicates what change may take place in the reflection coefficient of the mirror after it has been installed, without necessitating a new candle-power scale. The results of this investigation are given in Table II. A study of this table shows that if the reflection coefficient of the mirror is 0.80 or 0.90 instead of 0.85, i. e. if it is 5 or 6 per cent different from that used in computing the scale, the extreme error which occurs at the lower limit of the scale, i. e. at $J = 2.15$ cp, is only 0.3 per cent. Even if the reflection coefficient is 10 or 12 per cent different, the error is negligible except at the extremely low value of $J = 2$ cp where it may reach 0.7 per cent. At this position of the photometer, however, larger errors may occur from other sources.

In the above discussion it was implied that the substitution method of measurement is employed. Since this method is almost universally used there is no need of considering any other. In this method a standard 16-cp lamp is placed at O , and the voltage on the comparison lamp at O' is changed until a balance is obtained at the 16-cp point at the middle of the bar. In the employment of an auxiliary mirror the illumination is almost doubled, so that the comparison lamp must have an intensity of 25 or 30 cp, being different for mirrors of different reflection coefficients. If, however, the comparison lamp is made to balance the 16-cp standard at the middle of the bar, no matter what the reflection coefficient may be, the readings on a scale computed in terms of $r = 0.85$ will be correct near the 16-cp point, and the error at 2 cp will only be 0.7 per cent provided the reflection coefficient is not more than 10 or 12 per cent different from 0.85.

It has been assumed in the above discussion that the reflection coefficient, r , remains constant for all positions of the photometer screen. There are two reasons, however, why this may not be true: (1) the mirror must be approximately plane, and must have approximately the same coefficient of reflection at different parts of the surface in the region employed; (2) it must be shown that the angle of incidence corresponding to different positions of the

photometer screen does not change sufficiently to produce a difference in r . Measurements made on plate glass mirrors indicated that the error due to the latter cause would probably not be larger than 0.3 or 0.4 per cent, even over the entire range of the scale. Moreover, both this error and that which would result if the mirror were not plane or uniform become entirely negligible if the photometer is always kept near the center of the scale. This is accomplished in the photometer at the Bureau by the use of a rotating sector disk, calibrated for 32, 50, 64, and 100 cp, with corresponding candle-power scales on a revolving drum. To obtain values lower than 16 cp, such as 8 and 4 cp, the disk is placed on the other side of the photometer, between it and the comparison lamp. This arrangement was employed before the investigation of the auxiliary mirror was undertaken as being much less liable to error than an extended range of candlepower values on a single scale.

With reasonable care in the selection of the mirror, however, there is no need to fear any large error due to any of the above possible sources of error even if a single scale is used. The worst errors that could occur would be those in measuring lamps of 2 or 3 cp, and since such measurements are seldom required, and the accuracy demanded is never high, the single mirror method can be used with entire confidence.

PURITY AND INTENSITY OF MONOCHROMATIC LIGHT SOURCES.

By P. G. Nutting.

Spectrum lines differ greatly in spectral width or purity even when produced under similar conditions. An increase in intensity from any cause or an increase in density of the radiating gas always causes an increase in width or decrease in purity of a line. The choice of a monochromatic source for a given piece of work will depend upon (1) the maximum of purity and minimum of intensity required for the work in hand, (2) the conditions of production under which a given line will be as bright as possible for a given width, and (3) upon what lines, under given conditions, will afford a minimum of impurity with a maximum of intensity.

In polarimetry, for example, great intensity is required, while excessive purity is not essential in measuring ordinary rotations. Interferometry, on the other hand, requires excessive purity, but only a moderate intensity. This paper is devoted to a consideration of conditions (2) and (3) under which a maximum ratio of intensity to purity may be obtained.

A. A. Michelson¹ and F. L. O. Wadsworth,² from the "visibilty" of lines in the interferometer, calculated their width and established relations between spectral width and atomic weight and pressure. This method of determining width is free from corrections for slit width and resolving power, but involves considerable uncertainty in the matter of difference in path or time phase at the source. With both the Michelson and Fabry-Pérot interferometers the fainter components of composite lines are easily overlooked and ignored.

¹ Phil. Mag., (5) 34, p. 280-299; 1892.

² Astrophysical Journal, 6, p. 30; 1897.

Many lines are included in the published data of Michelson and Wadsworth which are known to be composite. The author³ has measured the width of lines directly with an echelon provided with a micrometer eyepiece. Corrections for slit width and resolving power were determined experimentally and applied. With the echelon the character of a composite line is at once evident.

THE THEORY OF SPECTRAL PURITY.

The mathematical definition of the spectral purity of a light source is more or less arbitrary. For instance, the total width of a line in wave length units might be taken as a measure of purity. This would be analogous to an ordinary length standard defined as the distance between two scratches on a bar. If one of the scratches is a broad groove, the length is uncertain by the width of this groove. Suppose the green cadmium light waves magnified to the length of such a meter bar. Then one of the limits would be three scratches, the outer ones 0.01 mm apart. On this basis the light is pure or the length is certain to one part in 100,000. On a similar basis sodium yellow is pure, at moderate intensity, to one part in about 700, mercury green to one part in 10,000, helium yellow to one part in 200,000 or one part in 15,000, if the satellite is included.

Such a definition of purity does very well, perhaps, for the sodium yellow, but is hardly fair in case the impurity consists of faint distant satellites. Obviously every component or component element should be weighted according to its intensity. Such a definition of purity leads mathematically to an expression in which impurity is analogous to the mean square error in the theory of errors and to the radius of gyration in mechanics. Hence corresponding to the center of mass in the latter case is a mean wave length which may be called the center of luminosity. Call this center of luminosity λ_c and represent by E the intensity of a component of wave length λ . Then

$$\lambda_c \equiv \frac{E_1\lambda_1 + E_2\lambda_2 + \dots}{E_1 + E_2 + \dots} = \frac{\int \lambda dE}{\int dE}$$

³ *Astrophysical Journal*, 24, p. 111-124; 1906.

by definition. The impurity I is defined by

$$I^2 \equiv \frac{E_1(\lambda_1 - \lambda_c)^2 + E_2(\lambda_2 - \lambda_c)^2 + \dots}{E_1 + E_2 + \dots} \\ = \frac{\int (\lambda - \lambda_c)^2 dE}{\int dE}$$

expressed in wave length units. It is frequently preferable to use a pure number, the *specific impurity*, which we may call I_s and define as

$$I_s = I/\lambda_c$$

Consider now the impurity of sodium yellow on this basis. For a pair of lines

$$I = (\lambda_1 - \lambda_2) \frac{\sqrt{E_1 E_2}}{E_1 + E_2} = (\lambda_1 - \lambda_2) \frac{K^{\frac{1}{2}}}{1 + K} .$$

where $K \equiv E_1 : E_2$, the ratio of the intensities of the two lines. For sodium yellow $K = 1.6$, $\lambda_1 = 5890.2$, $\lambda_2 = 5896.2$, hence $I = 6.0 \sqrt{1.6} : 2.6 = 2.9$ t-m. The specific impurity is $2.9 : 5892.5 = 0.00049$ or about one part in two thousand.

For a single line bounded by the limiting wave lengths λ_1 and λ_2 and uniform intensity throughout, $dE = h d\lambda$,

$$I^2 = \frac{\int_{\lambda_1}^{\lambda_2} (\lambda - \frac{1}{2}(\lambda_1 + \lambda_2))^2 h d\lambda}{\int_{\lambda_1}^{\lambda_2} h d\lambda} \\ I = \frac{\lambda_2 - \lambda_1}{2\sqrt{3}} = 0.2887\delta$$

where δ is the width of the line or band in wave lengths. A green line at 5500 measuring 0.055 t-m broad is thus impure to the extent of 0.016 t-m or to one part in three million. A complete continuous spectrum of constant energy throughout would have infinite impurity.

If the distribution of energy in a line or band is a cosine function throughout a total width δ , the impurity may be shown to be 0.217 δ .

Normal spectrum lines appear to possess an energy distribution corresponding roughly to the probability function

$$E = A e^{-a(\lambda - \lambda_c)^2}$$

where E is the energy per unit wave length.

This gives an integrable form for the impurity which comes out in this case simply

$$I = (2a)^{-1}$$

This case is of extreme importance. It represents not only most normal lines and many transmission bands but the luminosity curve of white light as well. If the distribution of energy throughout a spectrum is uniformly constant, then viewed by a mean normal eye, the distribution of luminosity is given by

$$L = L_0 e^{-\pi(\lambda - \lambda_c)^2}$$

λ being expressed in seventh-meters and L in light units per unit wave length. Hence the luminous impurity of "white" light is $1/\sqrt{2\pi} = 0.399$ s-m, and the specific luminous impurity is 0.08.

It is of interest to compare the impurities of lines of the above three types of structure having the same total energy and the same specific intensity at their centers, that is lines whose graphs have the same area and the same height.

Since in the three cases

$$E_1 = \text{const}(\equiv h), \quad E_2 = h \cos(\lambda - \lambda_c), \quad E_3 = h e^{-a(\lambda - \lambda_c)^2}$$

the condition of equal total energies gives

$$\delta_1 = \frac{2}{\pi} \delta_2 = \sqrt{\frac{\pi}{a}}$$

δ_1 being the width of the line of constant intensity throughout and δ_2 that of the line having a cosine distribution. The impurities in the three cases being as deduced above

$$0.288\delta_1, \quad 0.217\delta_2, \quad (2a)^{-1}$$

they are in the proportion

$$1, \quad 0.759, \quad 1.382.$$

The spectral width of a line of the third type measured visually is, however, quite different from the width $(\pi : a)^{\frac{1}{2}}$ given by equivalent energy. The visual impurity of lines of this type will be taken up later on.

ACTUAL AND VISUAL LINE WIDTH AND STRUCTURE

The brighter portions of a line or band always appear relatively less bright than they really are, for by Fechner's law the visual sensation of brightness increases with the logarithm of the actual luminosity. Hence the spectral curve of brightness is always flatter than the curve of luminosity and becomes zero when the intensity reaches the threshold value for the retina and wave length in question. Hence it is evident that impurity of radiation, luminous impurity, and visual impurity are in general quite different from one another. Further, a line may appear visually sharp and well defined, permitting of an accurate measurement of a definite width, even though its energy curve has wide, diffuse wings.

For example, consider an ordinary normal line whose energy curve $E(\lambda)$ is

$$E = E_m e^{-a(\lambda - \lambda_c)^2}$$

Its luminosity curve $L(\lambda)$ is $L = E : E_o$ or

$$L = \frac{E_m}{E_o} e^{-a(\lambda - \lambda_c)^2}$$

E_m being the intensity of the line (in watts per unit wave length) at its center and E_o the threshold value of the observer's eye for the wave length in question. Hence the curve of visual brightness $B(\lambda) \equiv \log L$ is

$$B = \log (E_m : E_o) - a(\lambda - \lambda_c)^2,$$

a parabola axial on the axis of ordinates. Hence such a normal line will have a central maximum $\log (E_m : E_o)$ and a definite apparent width

$$\delta\lambda = 2 \left(\frac{\log (E_m : E_o)}{a} \right)^{\frac{1}{2}}$$

extending from $B=0$ on one side of the center to $B=0$ on the other side. The first derivative of B has a large value, namely $\pm 2 (a \log E_m \cdot E_o)^{\frac{1}{2}}$ at the apparent edges of the line, hence such lines appear sharply defined.

It has been customary, in measuring the width of lines, to tacitly assume that the exponential energy curve is identical with or at least proportional to the curve of visual brightness. The resultant error is of slight consequence on rough work but in precise measurement assumes vital significance.

Analogous to radiant and luminous impurity, the visual impurity may be defined as the square root of

$$\frac{\int -\lambda_c)^2 dB}{\int dB}$$

in which the visual brightness $B(\lambda)$ is the logarithm of the luminosity $L(\lambda)$, which in turn is the ratio of the spectral energy $E(\lambda)$ to the threshold value $E_o(\lambda)$. Taking the above used exponential form for the energy curve of a line, its visual impurity as above defined may be calculated to be

$$I = \frac{1}{\sqrt{5}} \left(\log \left(\frac{E_m \cdot E_o}{a} \right) \right)^{\frac{1}{2}}$$

which is 0.223 of the (apparent) width of the line $\delta\lambda$.

NORMAL AND COMPOSITE LINES.

Classified according to structure and behavior, spectrum lines fall into two quite distinct classes, which may be distinguished as normal and composite. *Normal* lines are single and narrow when from a feebly excited source and with increasing intensity broaden continuously, reversing or doubling at some intermediate intensity. *Composite* lines consist of aggregates of other lines more or less fixed in position and relative intensity called primary and satellites but lying so close together as not to be resolvable with a prism or grating.

Composite lines are of many types, characterized by the form and variability of their components. The simplest are perhaps the yellow and blue helium lines. These have each a single companion line about half a tenth-meter removed from the primary on the side

toward the red, fixed in position and relative intensity. The yellow copper lines $\lambda 5700$ and $\lambda 5782$ are each usually double but sometimes triple. With increase in intensity, each component broadens *in situ* instead of retreating from its neighbor, as would be the case if the doubling or tripling were mere normal reversal. Triple lines are numerous in the spectral of cadmium, zinc, and related metals, component lines remaining fixed in position and relative intensity. The green thallium line divides unsymmetrically at one stage of its development, following which the broader component divides again, so that at one stage this line is triple in structure with all three components variable. The yellow sodium lines vary in a similar manner, becoming very complex before reversal proper occurs, the components blurring one another by overlapping to such an extent that it is difficult to tell just what does occur.

The seven chief visible mercury lines are all composite and all different in structure. They have been very successfully photographed by Janicki.⁴ Extreme types are $\lambda 4047$ consisting of four equidistant bands of about equal intensity and $\lambda 5461$ consisting of a strong primary with five faint satellites; two of these five are diffuse and very variable in position and relative intensity while the other three are extremely sharp and fixed in position, though variable in relative intensity.

These illustrations of compound structure indicate that its structure and variability should be carefully studied before a composite line is used at all as a monochromatic source. Only when its satellites are relatively faint and fixed or else very near the primary is a composite line to be used as a simple line. Let us calculate the impurity as above defined of Hg $\lambda 4047$.

Impurity of Hg 4047

Components	—0.05	primary	+0.06	+0.12 t-m.
Intensity	6	10	6	4
Impurity	$= (\Sigma E (\lambda - \lambda_c))^2 : \Sigma E)^{\frac{1}{2}} = 0.065$ t-m.			

Specific impurity $0.065 \lambda 4047 = 0.000016$ or about two parts in 100,000—an impurity prohibitive for the best work. With the green mercury line the case is different.

⁴ Ludwig Janicki: Inaugural Dissertation, Halle, 1905.

Impurity of Hg 5461.

Components	-0.09	-0.06	primary	+0.11	+0.13	+0.22
Intensity	1	2	40	8	2	5

Impurity (neglecting the width of the primary) 0.0067 t-m.

Specific impurity 0.00000125, or roughly one part in a million. This is sufficiently small to permit of the use of this line in interferometry. But the primary of this line alone has a width of about 0.04 t-m at moderate intensity, hence its impurity due to this width is nearly as great as that introduced by the presence of all the satellites together.

The brighter normal and composite lines are classified in the following brief catalogue. It is based on work with an echelon by the writer.⁵ The structure of the lines of zinc, cadmium, mercury, thallium, sodium, and hydrogen had been previously observed by Barnes,⁶ Houstoun,⁷ Janicki,⁸ and by several earlier workers.

Lithium, sodium, potassium, rubidium, and cesium.—The brighter lines of these spectra are all composite and much alike in structure, being characterized by variable overlapping satellites. All these lines appear single; that is, the satellites all sink below the threshold of vision at low intensities. With increasing intensity two, three, or four satellites appear, broaden *in situ* until they overlap and blur, until when reversal proper occurs the components have all merged in continuous wings. The fainter lines of these spectra have not yet been studied.

Magnesium, calcium, strontium, barium, gold.—These lines are either all normal, or if satellites exist they are so broad and overlapping as to appear as continuous wings. Some of the calcium lines show a trace of superposed structure similar to the lines of the sodium class.

Titanium, cerium, thorium, vanadium, chromium, molybdenum, tungsten, uranium, iron, cobalt, nickel, palladium, platinum, rhodium, and osmium.—All the brighter lines of these elements appear to be normal. They were all observed up to the point of reversal,

⁵ Astrophysical Journal, **23**, p. 64-79; **23**, p. 220-232; **24**, p. 111-125; 1906.

⁶ James Barnes: Astrophysical Journal, **19**, p. 190-212; 1906.

⁷ R. A. Houstoun: Philosophical Magazine, **7**, p. 456-467; 1904.

⁸ Ludwig Janicki: Ann. Physik, **19**, p. 36-79; 1906.

and had they been composite their structure would have been most apparent at about a tenth of that intensity. The fainter lines were not studied, but these being non-series spectra, there is every reason to expect that the fainter lines are also normal.

Manganese.—This spectrum is very remarkable. Twelve bright green lines are composite (double and triple) while all others are normal. As the arc fluctuates, the normal lines flash out and reverse while the composite lines, with their components fixed in position, remain as steady as though painted.

Copper.—The two yellow and two of the group of three green lines are composite (double or triple) while the third green line $\lambda 5104$ appears to be normal. The three composite copper lines are of a type that frequently occurs. One satellite is nearly as strong as the primary, of nearly constant relative intensity, and at a distance of about 0.1 t-m from it. The second satellite is on the opposite side of the primary, about 0.13 t-m distant, hardly a tenth as bright and very variable in relative intensity. Hence the same line may appear either double or triple under slightly different conditions. A similarly variable triple structure in cadmium $\lambda 5086$ has been shown⁹ to be the cause of the discrepancy between the wave length values determined by Hamy and by Michelson. Hamy's electrodeless tube gave a triple line, while in Michelson's tube with electrodes the second satellite was not apparent.

Silver.—All the five prominent silver lines are composite, $\lambda 5465$ being quadruple and the others triple, similar to the copper lines but more symmetrical.

Indium.—The prominent blue indium line appears to be normal, the doubling observed being a case of simple reversal.

Zinc, cadmium, mercury, thallium, tin, lead, and bismuth.—All the prominent lines of these spectra are composite. The zinc red and three blue lines are triple, resembling the composite copper lines. The red, green, and blue cadmium lines are similar triplets. The seven prominent visible mercury lines are all complex and all different in structure, with from two to five satellites. $\lambda 5769$ is a triplet resembling those of copper, cadmium, and zinc, but more compact. Thallium $\lambda 5350$ is a very peculiar triplet. Tin $\lambda 5631$

⁹ C. Fabry: *Astrophysical Journal*, 19, p. 116; 1904.

and $\lambda 4525$ are close triplets, lead $\lambda 6002$ and $\lambda 4058$ are broad, diffuse triplets, bismuth $\lambda 4722$ has two sharp, bright satellites and one faint and diffuse.

Aluminum, boron, beryllium, carbon, silicon, phosphorus, arsenic, and antimony give no visible lines of sufficient intensity to permit of a visual study of their structure.

Oxygen, nitrogen, sulphur, selenium, tellurium, chlorine, bromine, and iodine possess only faint visible lines in their primary spectra, while the lines of their secondary spectra are all over 0.2 t-m broad, and hence useless as monochromatic sources. The lines of the hydrogen primary spectrum are likewise too faint, while the secondary lines are not only broad but complex.

Helium has seven prominent visible lines that promise great usefulness as sources. Four of them have companion lines about half a tenth-meter distant, it is true, but these are relatively so faint as to be barely perceptible in an interferometer. The fact that a helium lamp improves rather than deteriorates with use is an immense advantage. All the helium lines are fairly narrow at moderate intensity (0.05 to 0.09 t-m). The relation of width to intensity will be taken up farther on.

Argon, neon, krypton, and xenon have not yet been investigated as to line structure and give little promise of usefulness, with the possible exception of the xenon red line.

There are then available two quite distinct classes of monochromatic sources, one class of fixed and the other of arbitrary purity.

1. Normal lines whose purity may be increased almost indefinitely by decreasing the density of the luminous gas and the intensity of its excitation.

2. Composite lines whose purity is high but in general can not be varied greatly. If all components remained fixed in position and relative intensity the purity could not be varied at all. Visual purity may always be increased by a decrease in intensity sufficient to lower the luminosity of fainter components below the threshold value.

The purity required of monochromatic sources for various purposes is roughly as follows. For *ultimate length standards* the impurity should not exceed one part in ten million, while the center of luminosity should be known to one part in one hundred million. It is very doubtful whether spectroscopists will ever be

able to produce a source of greater purity than this or whether other sources of error, for instance, those due to temperature variations, will ever be reduced to such an extent that greater purity will ever be necessary.

Interferometry requires ordinarily a purity of one part in one million and a knowledge of the center of luminosity to one part in ten million. Reference *standards of wave length* should be pure to one part in five hundred thousand and permit of the measurement of the position of their centers to one part in five million or, roughly, 0.001 t-m. *Refractometry* and *polarimetry* require only a low order of purity. One part in ten thousand is not too great an impurity for ordinary work.

To obtain the highest obtainable purity, then, it is necessary to choose from normal lines. Obviously the best normal lines to choose are those which (1) have the least minimum width and (2) which increase in width least rapidly with increasing intensity and density of the radiating gas. It was to determine which normal lines have the least minimum width and intensity coefficient that the following investigation was undertaken.

THE IMPURITY OF NORMAL SPECTRUM LINES.

Since the width of a normal line varies greatly with its intensity, apparatus was so arranged that the intensity of each line could be determined by a spectrophotometer at the same time that the width was being measured with a micrometer echelon. A wide slit near the source screened off all of it except the central portion, which was radiating toward the echelon. A Nernst lamp served as reference standard. This was eventually calibrated by means of a Rubens thermopile placed in the ocular of the spectrophotometer.

The width of a normal line appears to depend, aside from the pressure, upon the intensity alone. When, for example, copper is fed into an arc with carbon electrodes, the line 5106 is of the same width at a given intensity as when copper electrodes are used. It is the same whether metallic copper or a copper salt is fed into a carbon arc. If, as the salt burns away, the current is increased to bring the intensity of a line up to its original value, its width also is restored. Nor is this ratio of width to intensity appreciably affected by the addition of an alien salt or metal to an arc. A line

has less diffuse wings when produced in an oxyhydrogen flame than when produced in an arc, but it appears to possess the same mean width at a given intensity in the two cases.

Slit Reading	Line Width	Slit Width
5.4	121 μ	104 μ
5.3	110	94
5.2	98	84
5.1	91	74
5.0	79	64
4.9	67	54
4.8	53	44
4.7	39	34
4.6	26	24
4.5	15	14
4.4	(faint)	4
4.36	slit closed	0

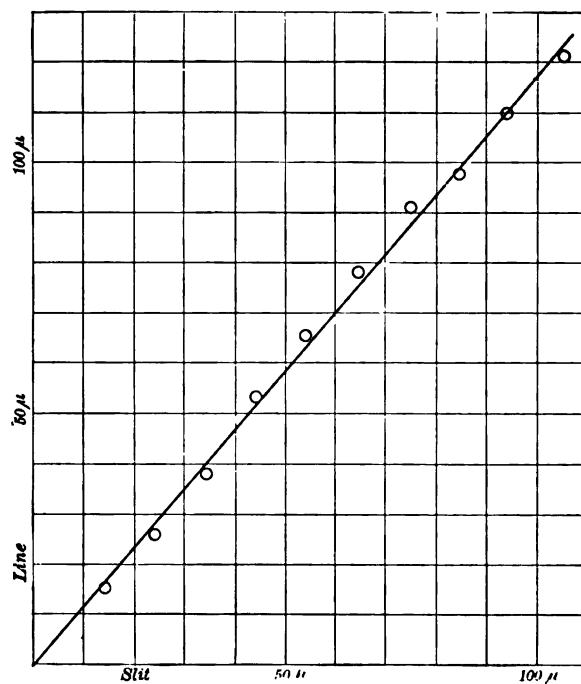


Fig. 1.

The green mercury line has three satellites on the side of the primary toward the red. These three lines are the narrowest bright lines known to the writer, and the inner one of these was for this

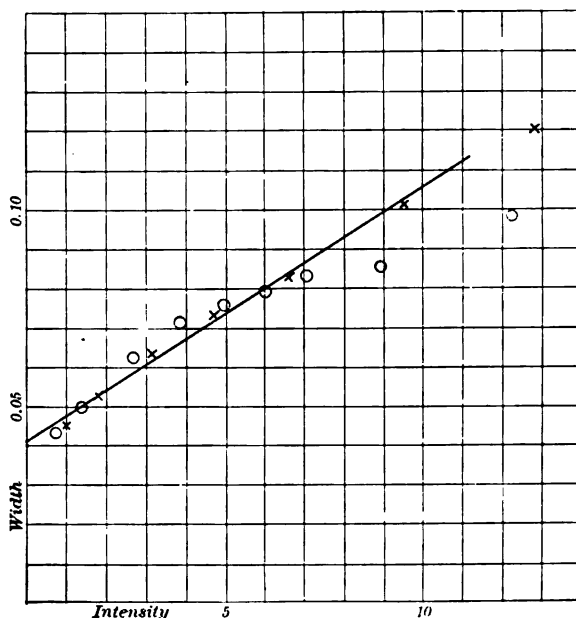
reason chosen to verify the correction for slit width to be applied to the measured width of a line. The figures quoted above and the points on the accompanying graph are from the first set of readings taken.

Line width and slit width are seen to be proportional to each other far within the possible errors of measurement. The resolving power of the echelon is seen to be considerably (at least five times) greater than the calculated value of 0.01 tenth-meter. The actual width of the line itself is from the plot certainly not greater than one micron or almost 0.001 t-m in terms of wave length. In other words, the light of this satellite of the green mercury line is homogeneous to within one part in at least five million. The main primary of this line is fully twenty times as broad as this satellite while the whole line, measured from extreme satellites, is three hundred times as broad or about 0.3 t-m.

The results of some observations on the relation of width to intensity of the yellow helium line $\lambda 5876$ are given below. They are the first and last of a series of observations. The helium was contained in short, stout Plücker tubes excited by means of a 5,000 volt transformer. The intensity was varied by varying the current in the primary. The two sets of observations quoted were made more than a week apart on two different Plücker tubes. The echelon slit was open 14 microns in one case and 24 microns in the other, the widths of the line being corrected by means of the plot reproduced in Fig. 1 above.

Tube 1, Slit Width 24 μ			Tube 2, Slit Width 14 μ		
Intensity	Line Width		Intensity	Line Width	
	Linear	Wave Length		Linear	Wave Length
1.0	42 μ	.046 t-m	0.85	40 μ	.044 t-m
1.9	47	.052	1.44	46	.050
3.2	57	.062	2.6	56	.062
4.7	66	.072	3.9	64	.071
6.7	76	.084	5.0	69	.076
9.5	93	.102	6.0	72	.079
12.9	109	.120	7.1	75	.083
19.0	115	.126	9.0	77	.085
			12.2	90	.099
			20.2	115	.126

The width of the line is seen to vary considerably with its intensity, more than doubling within the range available. The relation between width and intensity is approximately linear for low and moderate intensities. At great intensities the width is, if anything, less than this proportionality would indicate. At the intensity called to the capillary of the tube was just commencing to burn and show sodium lines, and possibly this might account for the falling off from the linear relation at high intensities, but arc lines show



solved composite. With exceptionally favorable conditions I have seen a faint separation into two components. If the line had two components, each of zero width, this limiting width of 0.040 t-m would represent the distance between their centers.

A few normal arc lines are sufficiently isolated in their spectra and variable in intensity to permit of fairly accurate observations of the width-intensity relation being made. Arcs of from two to eight amperes were used. As the arc fluctuated the maxima and minima were found to be reasonably constant in intensity and to permit of measurement. The best of the results are given below in the form of linear equations between width W and intensity I . They are taken from the plotted curves, all of which were of the same general shape as that reproduced in Fig. 2. W is expressed in tenth-meters and I in terms of an arbitrary unit such that the intensity of the bright green line Fe $\lambda 5328$ from an iron arc of 5 amperes in open air is about 0.5.

Ba	6342	$W = 0.034 + 0.0030 I$
Ni	5477	$W = 0.032 + 0.0035 I$
Fe	5328	$W = 0.025 + 0.0045 I$
Cu	5106	$W = 0.033 + 0.0030 I$
Fd	5296	$W = 0.032 + 0.0040 I$

for comparison may be added

He	5876	$W = 0.040 + 0.007 I$
Hg	5461 satellite	$W \quad 0.001$

The results for the arc lines show a striking uniformity both in limiting width and in intensity coefficient. Other lines gave data so similar and the accidental errors were so large that it was found to be only duplicating work to take measurements on them and only minimum width and intensity at reversal were recorded. With the arrangement used (echelon slit *perpendicular* to the spectrum lines formed by the preliminary dispersing apparatus) many lines were simultaneously in the field of view, so that if any of the many hundred lines observed had possessed exceptional qualities they would have been at once noticeable.

The minimum width of all normal arc lines produced at atmospheric pressure lies between 0.02 t-m and 0.05 t-m. This width is of the same order as the width to be expected from the Doppler effect and the kinetic gas theory. For, if δ is the spectral width of a line produced by a source moving with a mean speed of u cm/sec, then if V is the velocity of light, $\delta:\lambda = 2u/V$. For air at room temperature and pressure, u is usually taken as about 4.5×10^4 cm/sec, increasing with the square root of the temperature. Hence the width of a green line should be at least $\delta = 5000 \times 2 \times 4.5 \times 10^4 \cdot 3 \times 10^{10} = 0.015$ t-m. If the arc gases are at a temperature of 3000° , then this width should by the kinetic theory be multiplied by the factor $(3000/300)^{1/2} = 3.16$, hence $\delta = 0.015 \times 3.16 = 0.047$ t-m. Similarly for red line at 7000 , $\delta = 0.034$ t-m.

Reversing the argument, the widths observed indicate that only the thermal motions of the radiating particles are concerned in the broadening. The conduction of a current by the gas in the arc does not add greatly to the translatory motions which the particles of a gas would have in virtue of their high temperature alone. In other words, line-of-sight motions are unaffected by the current through an arc.

At low pressure an enclosed arc with electrodes of iron, copper, brass, and nickel were used. With nickel some observations were obtained with a clear window without disturbing the photometric adjustments used in the previous work. For Ni $\lambda 5477$,

$$W = 0.006 + 0.007 I$$

with a large probable error on account of the great fluctuations and altered character of the arc flame. Other lines were observed to have roughly the same minimum width. From kinetic theory the width at 1 cm pressure should be $(1/76)^{1/2} = 0.11$ of the width atmospheric pressure if arc vapors remain at constant density. The latter for $\lambda 5477$ at 3000° is 0.043 t-m, hence at 1 cm pressure should be 0.0047 t-m, considerably less than the width observed.

The development of a typical normal line is then somewhat as depicted in Fig. 3, where frequency (or wave length) is plotted against intensity. At high pressures, say in an open air arc, the development of the line is shown by the full lines, at low pressures

by the dotted lines. The intercept ab on the frequency axis represents the minimum width of impurity of the spectrum line at a very low intensity. With increase of intensity the line broadens and finally separates into two as the intensity corresponding to the point c is reached. With further increase in intensity the two components continually broaden and separate. At low pressure the minimum width is less and the line twins at a lower intensity.

There appears, then, to be little choice between various normal lines so far as minimum width and intensity coefficient are concerned. Normal lines certainly do not differ by as much as a factor of ten in these properties. There being so little diversity leaves a wide range of lines equally available. The green nickel line is perhaps as easily produced and segregated as any other.

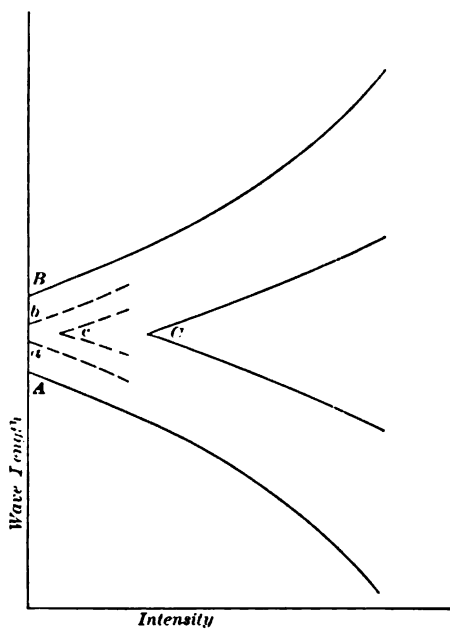


Fig. 3

If a number of lines of the same energy intensity, but in different parts of the spectrum are available, it is of interest to know which will be best for visual use. In this case their relative luminosity will be given roughly by the function,

$$V = e^{-\pi(\lambda - 5.1)^2}$$

for an average observer, and their relative visual brightness by the logarithms of these values. This function has been deduced from a great number of observations by different investigators. Numerical values of this function are given below:

$\lambda =$	510	520	530	540	550	560	570	580	590	600	610	650 $\mu\mu$
	510	500	490	480	470	460	450	440	430	420	410	360 $\mu\mu$
$V =$	1.00	0.969	0.882	0.753	0.611	0.455	0.324	0.224	0.133	0.078	0.041	0.0008

In conclusion, the character of some of the purest known sources of illumination may be summarized:

1. The most nearly homogeneous light sources yet studied are some of the satellites of the composite lines. Three satellites of the green mercury line are homogeneous to within one part in at least five million. These are, however, not yet available for interferometry because their segregation involves the use of an echelon grating. These satellites appear to be free from the Doppler effect, but this point is not yet well established.

2. Second in homogeneity at moderate intensity are the normal arc lines. These are all roughly 0.05 t-m broad at atmospheric pressure, and hence possess a specific impurity of about one part in five-hundred thousand. Produced at low pressures these normal arc lines are approximately 0.006 t-m broad, with a specific impurity of one part in four million.

3. Of third rank as to purity are the composite lines produced in vacuum tubes measured between extreme components. These have a width varying from 0.1 to 0.3 t-m, or even more.

The purest of these, namely, the zinc and cadmium lines, have a specific impurity (as defined in this paper) varying from one part in a million to one part in sixty thousand.

RADIOMETRIC INVESTIGATIONS OF INFRA-RED ABSORPTION AND REFLECTION SPECTRA.

By W. W. Coblentz.

INTRODUCTION.

In the determinations of the physical properties of matter, such as electrical and thermal conductivity, specific inductive capacity, coefficients of expansion and refraction, etc., the spectroscopic study of the transparency of substances to radiant energy and in particular to heat waves, has been left in the background, although this data is just as necessary in meteorology, geology, and allied sciences as is our knowledge of any of the other so-called constants of nature.

The following research was undertaken to supply some of these needs, particularly to ascertain the cause of absorption, and the connection between absorption and the structure of crystals. In this and in a previous research it has been shown that certain absorption bands in the infra-red are due to certain groups of atoms. That these results have a bearing on the question of structure of crystals is obvious. For, if the crystal is composed of molecules of water and calcium sulphate, for example, which separately have characteristic absorption bands, then, if these molecules or certain groups of atoms in them undergo no physical change when they combine to form a crystal (of selenite in this case) then one would naturally infer that the absorption spectrum of the product will be the composite of the absorption bands of the two constituents.

This phenomenon is different from the one in which Julius¹ showed that the absorption spectrum of a chemical compound is *not*

¹ Julius: Verhandl. Koninkl. Akad. Amsterdam, Deel I, No. 1; 1892.

the composite of the bands of the constituent elements, as in this case the molecule has been changed.

Just how these specific groups of atoms are placed with respect to the crystallographic axes it is impossible to determine.

A practical question in optics, viz, the possibility of finding material suitable for prisms for use in certain regions of the spectrum in which our present prism material is opaque or in which its dispersion is small, was also kept in view.

The reflecting power of metals was included to fill a gap left in the work of Hagen and Rubens done at the Reichsanstalt. The reflection from minerals composing the earth's crust has unexpectedly furnished data which will be useful to the meteorologist. It may even help clear up such an obscure question as the radiation from the moon. The determination of the data presented involved the use of one of the most sensitive heat measuring devices yet produced, viz, the radiometer. The region examined is a vast one compared with the visible spectrum, and has heretofore been little explored.

I.

ABSORPTION SPECTRA.

Many chemical compounds contain both oxygen and hydrogen, which, on applying heat, pass off in the form of water. That this water is not united so tenaciously as the other constituents is evident from the fact that in many instances it can be easily removed; many salts give up their water if exposed to dry air at ordinary temperature. From the circumstance that many of these compounds are crystalline the water is said to be present as "*water of crystallization*." The manner in which the water exists in the crystal is not understood. By some it is considered a part of the chemical molecule. By others it is thought that the molecules of water exist in their entirety among the other molecules which constitute the crystal.

On the other hand, the water which is given off only at a red or even white heat can scarcely be present in the compound in the same manner as the water of crystallization and is distinguished as "*water of constitution*." In this case the water is not supposed to exist as such in the mineral, but to result from the union of oxy-

gen and hydrogen or from the hydroxyl groups contained in the compound.

The present investigation of this perplexing question makes use of the results of previous work in which it was shown by others as well as by the writer that certain groups of atoms have characteristic absorption bands. Hence, if the oxygen and the hydrogen which enter into the composition of certain minerals as "water of crystallization" are united as they are in a molecule of water then one would expect to find the absorption spectra of such minerals to be a composite of the bands of water, and of the bands caused by

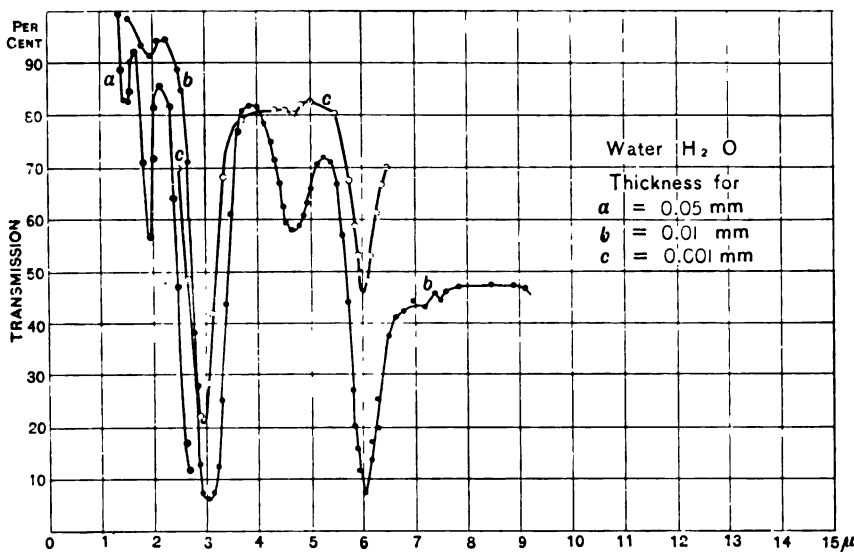


Fig. 1.

the other constituents. In other words, the criterion for distinguishing water of crystallization from water of constitution is the presence of the absorption bands at 1.5, 2, 3, 4.75, and 6 μ which have been shown to be characteristic of water. If there are no other absorption bands near by, then the intensity of these bands should be somewhat like that of the bands of water, viz, the bands at 1.5, 2, and 4.75 μ are weak, while the bands at 3 and 6 μ are very strong.

A hydroxyl group will also cause an absorption band at 3 μ . Silicates may have a small band shifting from 2.9 to 3.1 μ , but since it is weak there is no danger of confusing it with the strong water band in the same region.

APPARATUS AND METHODS.

In the present investigation a mirror spectrometer, a rock salt prism, and a Nichols radiometer were used. The mirrors were 10 cm diameter and of 50 cm focal length, while the rock salt prism was an unusually fine one, having faces 9×9 cm area. The methods of adjusting the apparatus and mounting the specimens before the

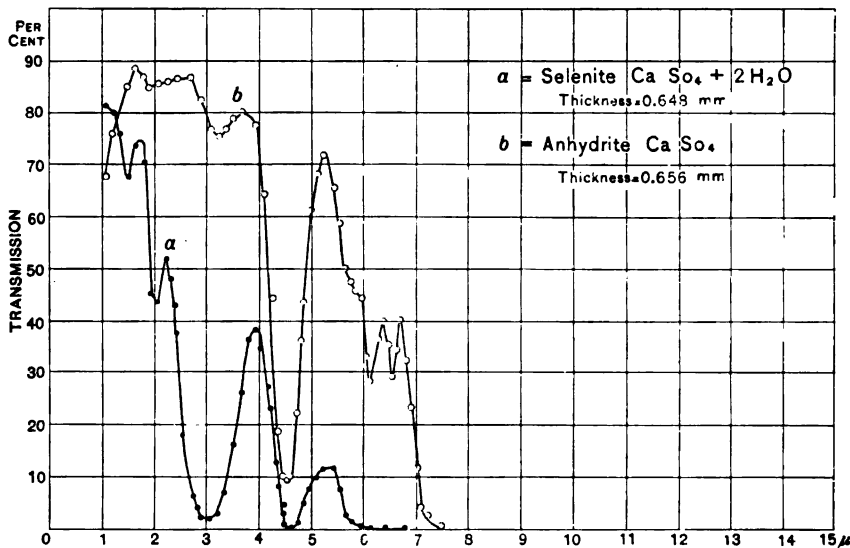


Fig. 2.

spectrometer slit was essentially the same as in a previous research² and need not be mentioned here. The spectrometer slits were 0.3 mm wide, or about $2'$ of arc.

Group I.—Minerals Containing Water of Crystallization.

Under this heading will be given absorption curves of a number of minerals containing water of crystallization. For various details in regard to the manner of cutting and mounting these minerals the reader is referred to the complete research which is being published by the Carnegie Institution.

² Investigations of Infra-Red Spectra, Published by Carnegie Inst. of Wash., 1905. Abstract in Phys. Rev., 20, p. 273; 1905.

Water, H_2O . First of all it will be necessary to consider the transmission curves of water, Fig. 1, in which are to be seen the characteristic absorption bands at 1.5, 2, 3.0, 4.75, and 6 μ .

Selenite, $CaSO_4 + 2H_2O$ and Anhydrite $CaSO_4$. Of all the minerals, containing water of crystallization, studied, these two most conspicuously demonstrate the effect of the presence of water. It would have been desirable to study more minerals in the hydrous and anhydrous state, but none were obtainable. The artificially dehydrated minerals, such as copper sulphate, were too opaque for examination after expelling all the water. The only exception is

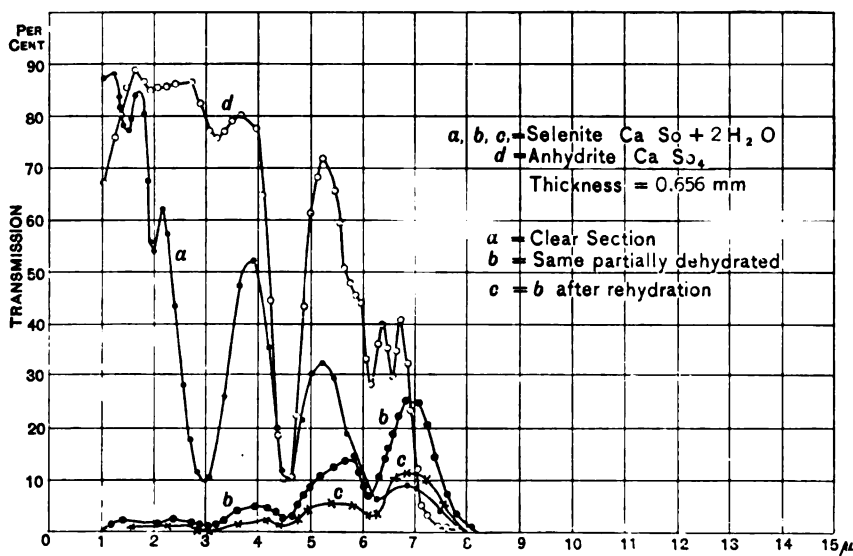


Fig. 3.

selenite, the various curves of which will now be considered. The anhydrite curve, b , Fig. 2, is to be noticed first, from which it will be observed that there are small bands at 1.9, 3.2, 5.7, 6.15 and 6.55 μ , and an enormous band at 4.55 μ , which will be shown later to be due to the SO_4 ion.

Turning to the selenite curve, a , Fig. 2, it will be noticed that it is less transparent, for the same thickness, 0.65 mm, and that in its general trend it is similar to the curve for water. All of its absorption bands coincide with those of water, with the exception of the 4.75 μ band, which is shifted to 4.6 μ . The shifting of this band

is due to the SO_4 band at 4.55μ , as was found on examining the anhydrite.

In Fig. 3 is illustrated the effect of dehydrating the selenite. This was accomplished with difficulty on account of the warping and shrinking of the plate, which necessitated the dismounting of the specimens for each heating. It was found necessary to clamp the specimen between two metal plates in order to prevent it from warping or breaking. Curve *a* gives the transmission of a clear piece having a thickness of 0.204 mm. Curve *b* shows the transmission after partial dehydration. It is of considerable interest in

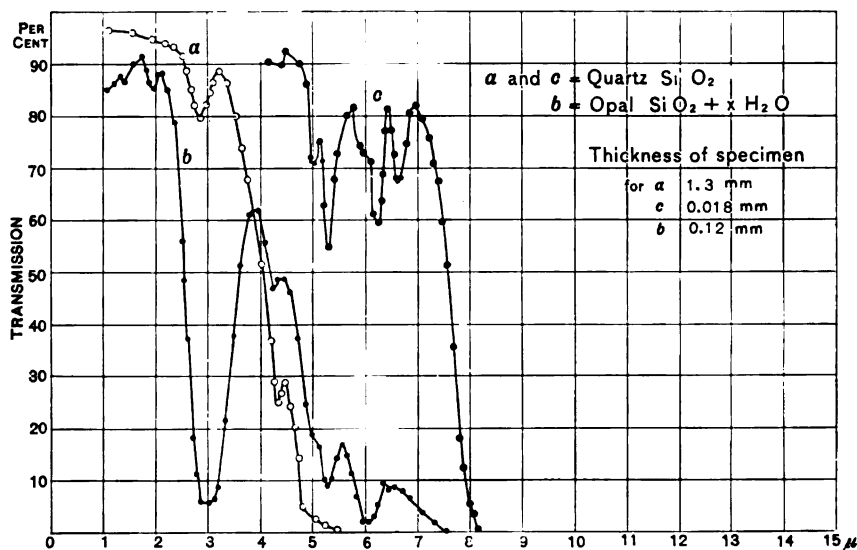


Fig. 4.

showing the permanence of the 4.55μ band. Beyond 6μ the transparency increases and becomes 1.5 times as great as selenite at 6.9μ . The dehydrated section, which is an opaque white mass, was then moistened with water and allowed to stand over night to dry and set, just as in the case of plaster of Paris. The result is shown in curve *c*, where the transparency at 6.9μ has decreased to almost the original value found for the transparent crystal. At 7μ the increase in absorption due to the presence of water can not be doubted.

Quartz, SiO_2 and Opal, $\text{SiO}_2 + x\text{H}_2\text{O}$. The transmission curve *a*, Fig. 4, of quartz shows small bands at 2.9μ and 4.35μ .

Opal is quartz containing variable proportions of water—from 5 to 30 per cent. It shows no traces of crystallization. The transmission curve of opal has the general outline of the curve for water and contains the bands of water at 1.5, 2, 3, and 6 μ , as well as the silicon bands at 4.2 and 5 μ . The 3 and 6 μ bands are the composite of the water and silicon bands in those regions.

Heulandite, $H_4CaAl_2Si_4O_{18} + 3H_2O$. In Fig. 5 two curves of heulandite are given. The sections were about the same thickness and the great difference in transparency is due to the fact that the crystals were of different homogeneity and transparency. The curves show the presence of the water bands at 1.5, 2, 3, and 4.75 μ , beyond which point the opacity becomes too great for further exploration.

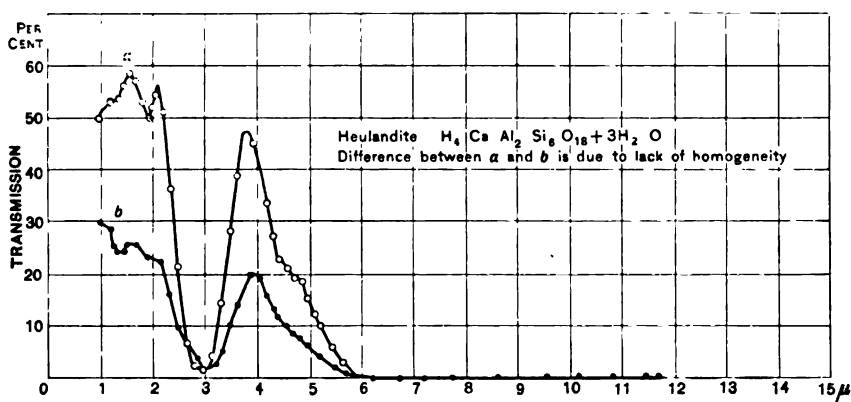


Fig 5.

Stilbite, $CaAl_4Si_6O_{18} + 6H_2O$. In Fig. 6 the curve of stilbite is to be noticed for its great transparency with all the water bands superposed. There do not appear to be any important bands belonging to the mineral itself.

Potassium Alum, $K_2SO_4 \cdot Al_2(SO_4)_3 + 24H_2O$. (Fig. 6.) The water bands at 1.5 and 2 μ are present. Owing to the 24 molecules of water present and SO_4 bands at 4.55 μ (to be mentioned later) it is not possible to penetrate beyond 3 μ unless a thinner section be made.

Natrolite, $Na_4Al_2Si_3O_{10} + 2H_2O$. The curve *a*, in Fig. 7 shows the water bands in their usual position and proper intensities.

Scolecite, $CaAl_2Si_3O_{10} + 3H_2O$. (Fig. 7.) This mineral differs

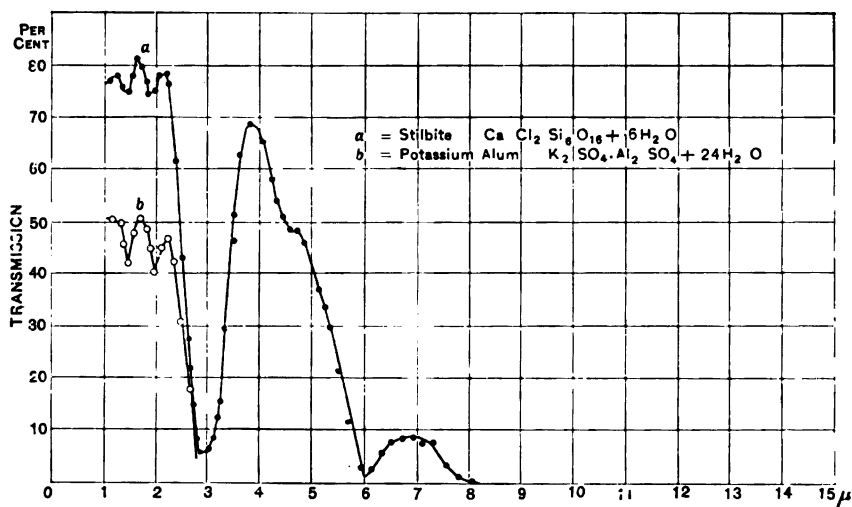


Fig. 6.

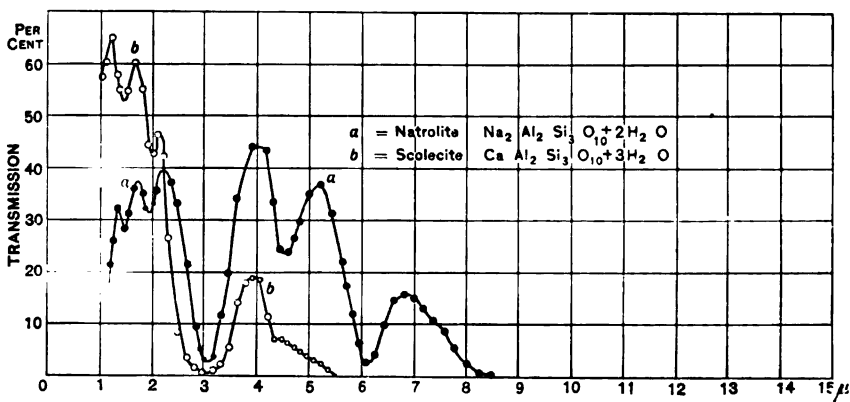


Fig. 7.

from the preceding in being a lime zeolite, and in having one more molecule of water. Its greater homogeneity makes it more transparent in the region of the short wave lengths. It has the general outline and the absorption bands of the water curve up to 5μ where it is completely opaque. The 3μ band is evidently complex.

Calcium Chloride, $\text{CaCl}_2 + 6\text{H}_2\text{O}$. (Fig. 8.) This compound comes in large hexagonal crystals. A specimen was melted between two plates of rock salt, and left over P_2O_5 for several days, until the edges became white from dehydration. This substance recrystallized in the meantime, and, when examined, showed the water

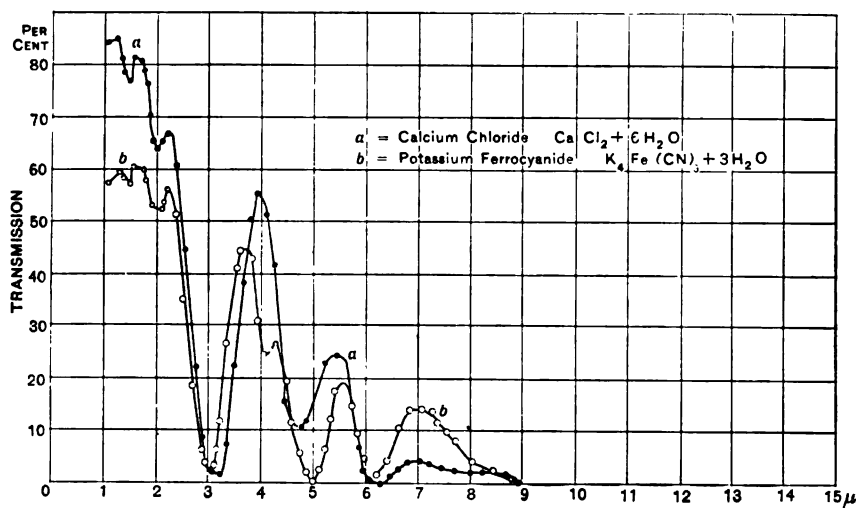


Fig. 8.

bands in their usual positions and intensities, except the 3μ band, which is shifted to 3.2μ .

Potassium Ferrocyanide, $\text{K}_4\text{Fe}(\text{CN})_6 + 3\text{H}_2\text{O}$. The transmission curve, *c*, Fig. 8, shows the water bands at 1.5 , 2 , and 3μ , while the next two bands are shifted to 5 and 6.2μ , respectively.

Group II.—Minerals Containing Water of Constitution.

The first part of this group contains minerals in which the oxygen and hydrogen are present in the form of the univalent hydroxyl radical OH , and which are known as hydroxides. The hydroxides when heated also yield water, but it is a characteristic that they

must be strongly heated before they are decomposed. From previous work, already mentioned, we would expect an absorption band at $3\ \mu$ for hydroxides.

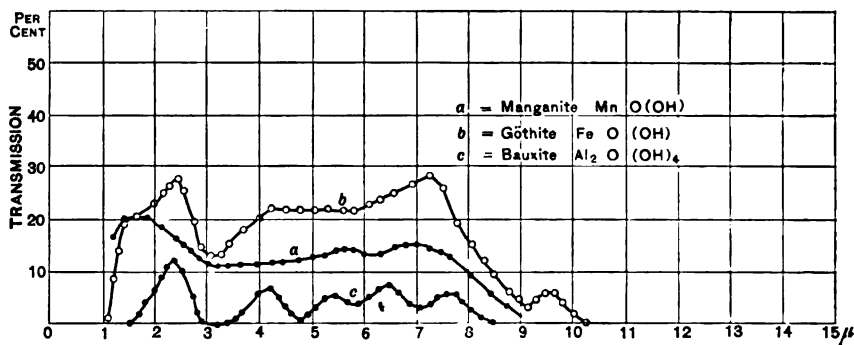


Fig. 9.

Manganite, MnO(OH) . As will be noticed in curve *a* of Fig. 9 manganite has no sharp bands throughout the whole region to $9\ \mu$. There is a slight indication of a band at $3\ \mu$ and another at $6.2\ \mu$.

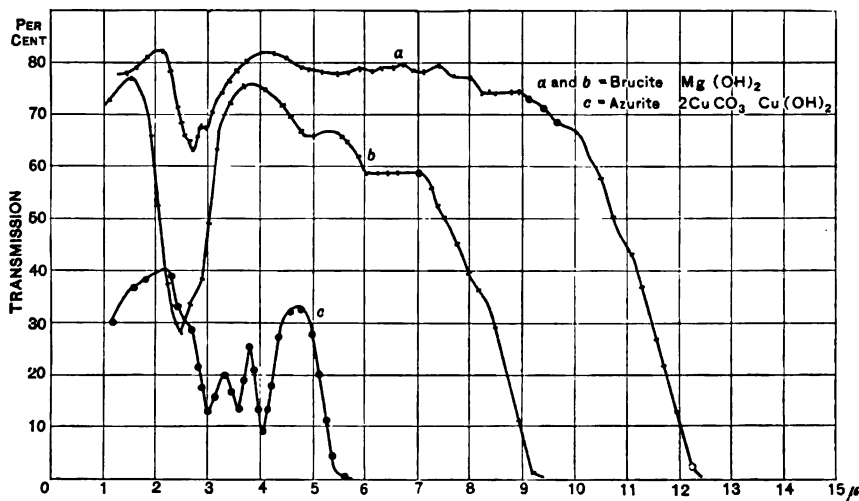


Fig. 10.

Göthite, FeO(OH) . In appearance this mineral is opaque and black with a metallic luster, and as a whole would not lead one to suspect the fact that it is fairly transparent to $9\ \mu$. It transmits about

20 per cent of the energy throughout this whole region, which is in marked contrast with the almost complete opacity of iron. The transmission curve is given in Fig. 9, *b*.

The $3\ \mu$ band is well marked, while a second one occurs at $9\ \mu$. This specimen was kindly presented by Prof. S. L. Penfield.

Bauxite, $Al_2O(OH)_3$. In this mineral the $3\ \mu$ band is wide and complex. There are other bands at 4.7 , 5.8 , and $7\ \mu$. Curve *c*, Fig. 9.

Portland Cement. (Curve *e*, Fig. 11.) The chemical constitution of Portland cement is unknown. According to LeChatelier³

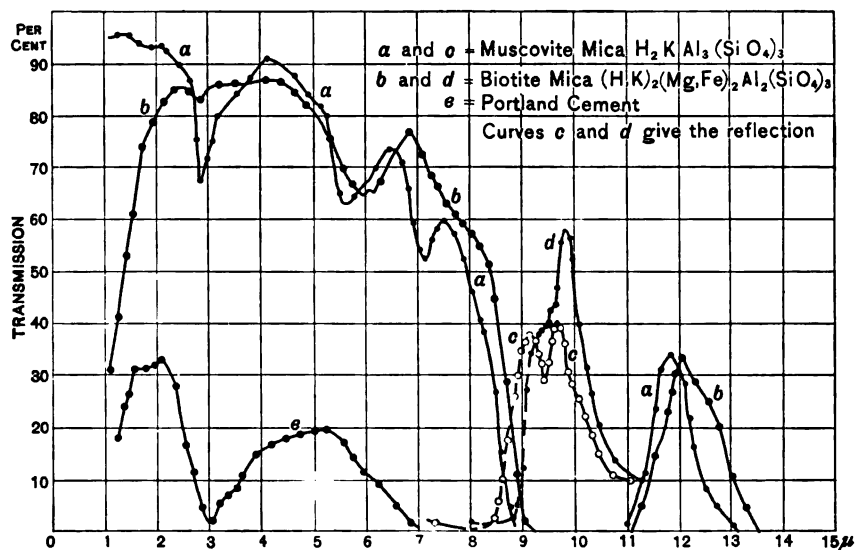


Fig. 11.

the action of water causes the formation of a hydrous silicate, $2(SiO_2 \cdot CaO) \cdot 5H_2O$, and a hydrous aluminate, $Al_2O_3 \cdot 4CaO \cdot 12H_2O$.

From the curve shown it will be noticed that the substance is very opaque, with a large absorption band at $3\ \mu$, which would seem to indicate hydroxyl groups. From the curves of mellite and alum it appears that Portland cement is too transparent beyond $3\ \mu$ to contain 17 molecules of water of crystallization, as indicated in the aforesaid formulæ.

Azurite, $2CuCO_3 \cdot Cu(OH)_2$. There are absorption bands at 2.5 ,

³ LeChatelier: Annales des Mines, Feb., 1888.

3.05, 3.58, and 4.05 μ and complete opacity beyond 5.5 μ . The 3.05 μ band is shifted slightly from the position common to substances containing OH groups. Curve *c*, Fig. 10.

Brucite, $Mg(OH)_2$. In Fig. 10, curves *a* and *b*, is shown the transmission through magnesium hydroxide. Curve *a*, which is for a very thin film (0.05 mm), shows the 3 μ band common to OH groups.

Micas; *Muscovite*, $H_2KAl_3(SiO_4)_3$; *Biotite*, $(HK)_2(MgFe)_2Al_2(SiO_4)_3$. The constitution of the micas is involved in a greater or less degree of uncertainty. They are all silicates of aluminum and either K, Na, Li, or of Fe and Mg. All the micas yield water upon

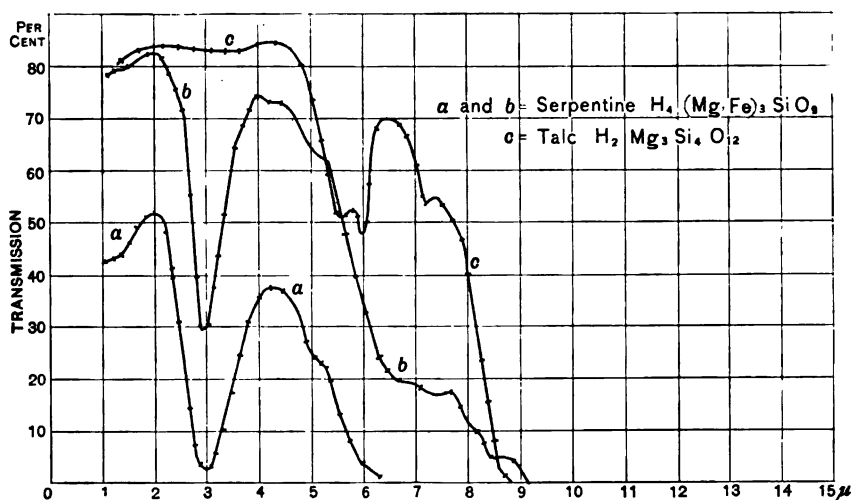


Fig. 12.

ignition, but it is uncertain whether this water of constitution is due to the presence of H or OH. There is no deep wide absorption band at 3 μ , so that, judging by the present method of examination, there can not be any hydroxyl groups present. Curves *c* and *d*, Fig. 11, are for reflection.

Serpentine, $H_4(MgFe)_3SiO_9$. In serpentine the water is chiefly expelled at a red heat. From the curves of serpentine which show a large absorption band at 3 μ , if the results from the present methods of examination are to be trusted, it would appear that there are hydroxyl groups present. From the results obtained by Clarke

and Schneider⁴ it was inferred that the Mg is present as MgOH. Curves *a* and *b*, Fig. 12.

Talc, $H_2Mg_3Si_4O_{10}$. In talc water is expelled at a red heat. From the investigation of Clarke and Schneider (loc. cit.) no hydroxyl groups were inferred, and the transmission curve does not show a band at $3\ \mu$. Curve *c*, Fig. 11.

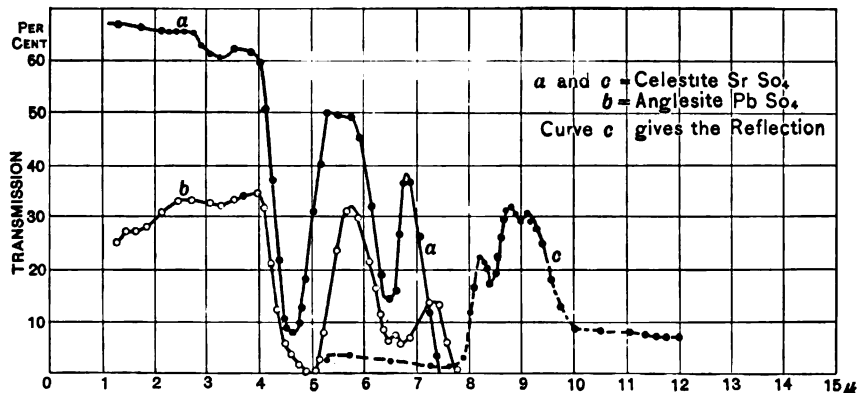


Fig. 13.

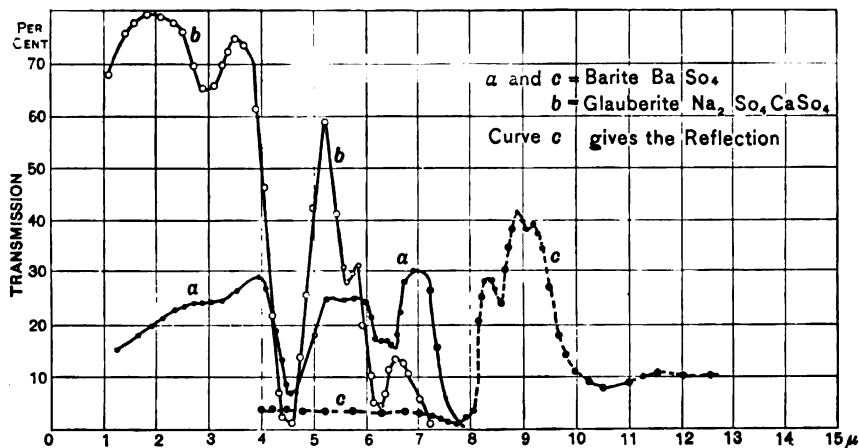


Fig. 14.

Group III.—Miscellaneous Compounds.

Sulphates. In Figs. 13 and 14 the transmission and reflection curves of the sulphates of strontium, lead, barium and sodium, and

⁴ Clarke and Schneider: Amer. Jour. Sci., 40, p. 308; 1890.

calcium are given. Space will not permit discussing them at length. All of them show harmonic bands at 4.55 and $9.1\ \mu$ as was noticed in anhydrite. The region of 6 to $7\ \mu$ is also characteristic of the sulphates, and as a whole there is sufficient evidence for attributing these bands to the SO_4 ion.

II.

REFLECTION SPECTRA.

Reflecting Power of Metals. (Fig. 15.)

The reflecting power of various metals and alloys which can be easily produced in the form of concave mirrors has been measured

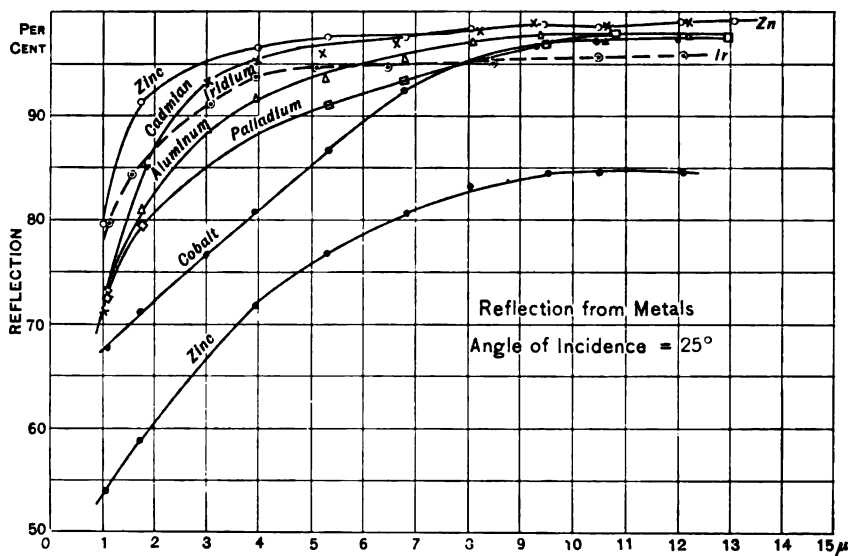


Fig. 15.

at the Reichsanstalt by Hagen and Rubens⁵. The list does not include cobalt, zinc, cadmium, aluminium, palladium, and iridium, the reflecting power of which in the form of plane mirrors is here-with presented.

The present list can not of course be of as great practical use as the metals previously examined, since the surfaces tarnish, but from

⁵ Hagen and Rubens: Ann. der Phys., 8, p. 1; 1902; Ann. der Phys., 11, p. 873; 1903.

a theoretical standpoint they are of considerable importance. For example, Hagen and Rubens established relations between the reflecting power and the electrical conductivity of the metals studied. One would therefore expect similar relations for closely related metals in the Mendeleeff's series. For example, one would expect the reflecting power of cobalt to be of the same order as that of nickel, and a similar relation between zinc and copper and palladium and platinum. From the present examination (see Table I) it will be noticed that such a close parallelism exists in all cases where the actual condition of the reflecting surface, i. e., its polish, is negligible. Unfortunately in the present list only zinc and cobalt take a high polish, which is quite permanent. Cadmium takes a high polish but tarnishes in a day or two. Tin can not be given a high polish; palladium is of a similar nature, while aluminum always retains a hazy white surface. As a result, in the shorter wave lengths the reflecting power is lower than the normal and rises steadily to 8 or 10 μ where it assumes a constant value which can no doubt be interpreted as real.

The specimens examined were about 3×4 cm area. They were ground plane then polished with Vienna lime and stearin oil. The standard silver mirror was finally prepared by "buffing," a fine surface being obtained.

An attempt was made to use silver on glass mirrors, but even those that transmitted only blue light were found to differ as much as 2 per cent in reflecting power while the best silver on glass mirror reflected about 0.5 per cent less than the one of pure silver at 5 μ to 12 μ . A mirror of pure silver was therefore used as a standard of reference. It consisted of a sheet of the pure metal, 0.5 mm thick, soldered on a heavy plate of brass.

The method of observation consisted in placing the standard silver mirror and the comparison mirror upon the carrier before the spectrometer slit, as in the preceding work, and obtaining the ratio of the deflections.

This gives the reflecting power relative to silver, and is slightly higher than the absolute reflecting power, since silver is not a perfect reflector. The absolute reflecting power of the metals, given in Table I, are found by multiplying the relative values by the reflecting power of silver, given in the first column of the same table.

TABLE I.
Reflecting Power of Metals.
(Absolute Values.)

Wave Length μ.	Silver (Massive) H. & R.*	Nickel (Massive) H. & R.*	Cobalt (Massive sheet)	Zinc (cast)	Cadmium (cast)	Aluminum (sheet)	Tin (cast)	Palladium (sheet)	Platinum H. & R.*	Iridium (Massive sheet)
1.06	96.4	72.0	67.5	79.4	70.8	73.3	54.0	74.8	72.9	79.4
1.71	97.3		71.5	91.0	85.0	80.8	59.3	79.3	79.5	84.7
3.06	97.3	88.7	76.7	95.5	93.0	88.3	68.6	87.5	88.8	91.4
3.96	97.7	92.5	80.7	96.2	95.2	91.4	71.7	88.1	91.5	93.3
5.24	97.3	94.7	86.2	97.2	95.9	93.8	76.7	90.4	93.5	94.2
6.75	98.5	94.8	92.7	97.2	97.0	95.2	80.3	93.3	95.5	94.7
8.02	99.0	95.0	95.8	98.0	97.8	96.9	83.2	94.7	95.1	94.8
9.38	98.9	95.6	96.4	98.1	98.4	97.4	87.0	95.3	95.4	95.6
10.49	99.0	95.8	96.8	98.4	98.3	96.9	87.0	96.6	95.9	95.8
12.03	98.9	95.7	96.6	98.3	98.2	97.3	86.9	96.5	96.5	96.1
13.00	99.0	95.6		99.0					96.4	

* Hagen and Rubens's values are designated by "H. & R."

The reflecting power of nickel and platinum are quoted from Hagen and Rubens (*loc. cit.*) to show their close relations with cobalt and palladium, respectively. The angle of incidence 25° was permissible, since it is well known that the reflecting power increases but slightly up to this angle. Of course, the assumption is tacitly made here that the change in reflecting power with angle of incidence is the same for all the metals examined, and since we are finding a ratio will therefore affect alike the numerator and denominator of the fraction. Any error thus introduced could be only a small fraction of a per cent, which is as accurate as the variation in the polish and planeness of different samples of the same metal will permit.

The low reflecting power of most of the metals examined in the region of $1\ \mu$ is due more to lack of polish than to a possible transparent region such as obtains in silver in the ultraviolet.

Palladium is lower in reflecting power than platinum; and it is barely possible that it would have a slightly higher value, if a better surface could be produced. The specimen was made by soldering a $0.1\ \text{mm}$ sheet upon a heavy plate of brass.

Considerable difficulty was experienced in casting homogeneous plates of cadmium. Success was finally attained by melting it in a thin copper mold. When cooled, the mold was torn off and the plate, $1\ \text{cm}$ thick, then filed and ground plane. In Fig. 15 it will be noticed that its reflecting power suddenly rises to a constant value beyond $5\ \mu$.

The sheet of cobalt was about $0.5\ \text{mm}$ thick and permitted considerable filing and grinding. However, it was found impossible to prepare a surface that was free from pores. This probably explains its deviation in reflecting power from that of nickel, out to $10\ \mu$, where it reflects more than nickel as it should, since its electrical conductivity is higher.

The aluminum was a sheet of commercial material. It took a high polish. Its reflecting power is unusually high beyond $10\ \mu$, and is known to be practically a perfect reflector for heat waves at $25\ \mu$.

Little can be said concerning tin. It was found impossible to give it a polish, although the melted surface on cooling was very

bright. From its electrical conductivity it ought to have a reflecting power of the order of nickel and platinum.

Zinc is the most interesting of all the metals studied. It takes an unusually high polish which is quite permanent. Its color is very peculiar. It seems to have a low reflecting power in the

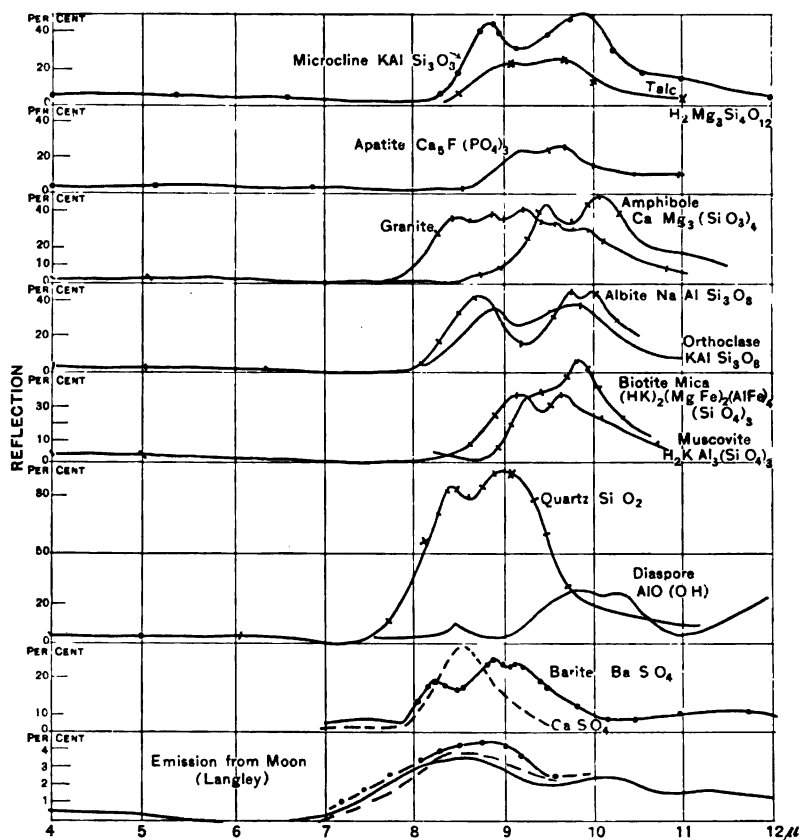


Fig. 16.

visible spectrum which rises suddenly to a maximum beyond 4μ , and in this respect compares favorably with silver, which is the highest and most serviceable reflector known for the visible and the infra-red spectra. The electrical conductivity, as well as the reflecting power of zinc and cadmium, are in close agreement. The metals, with the exception of aluminum, were from Kahlbaum and from

Heraeus. Pure cobalt is, of course, practically unobtainable, and the specimen examined probably contained from 1.5 to 2 per cent nickel. Hagen and Rubens (loc. cit.) have computed the absorption of the metals from the electrical conductivity by means of the formula

$$100-R = \frac{36.5}{\sqrt{c\lambda}},$$

where R is the reflecting power, c is the reciprocal

of the resistance of a conductor 1 m long and 1 sq. mm area, in ohms, λ = wave length in μ . They found a slight variation in the observed and computed values of $100-R$, the maximum being about 0.5 per cent at 12 μ . It must be said, however, that if they had selected the wave length $\lambda = 10.49 \mu$, where in many cases the value of R is frequently the same as for 12 μ , that the discrepancy would be larger and of the same order as that observed in the present results. In the present work no attempt was made to attain the accuracy of these two investigators for the reason that the nature of the material would not permit it. The difference in the observed and computed values of $100-R$ is given in the following table, using the values of the electrical conductivity as found by Jäger and Diesselhorst.⁶

The agreement in the observed and computed values (except for tin) is as close as one can expect from the nature of the metals examined.

TABLE II

$100-R$	Observed	Computed
Cd	1.8	2.87
Co	3.4	3.28
Pd	3.5	3.48
Zn	1.7	2.59
Sn	13	3.55
Al	2.7	1.89

Reflecting Power of Various Substances.

Previous work on this subject has been rather disconnected. Nichols⁷ compared the reflecting power of quartz with silver and

⁶ Jäger and Diesselhorst, quoted by Landolt and Börnstein.

⁷ E. F. Nichols: Phys. Rev. 4, p. 297; 1897.

found that, from the visible to 8μ in the infra-red, only a small amount is reflected, while at 8.5 to 9.02μ there are two intense reflection bands which have a reflecting power almost as high as that of silver. Various other substances have been examined by different investigators, but no extended series of chemically related substances like the silicates or sulphides have heretofore been investigated, although they are of great interest from a meteorological standpoint.

In Fig. 16 are shown graphically the unusual similarity of the reflection curves of various silicates which compose the earth's crust. From this it will be noticed that, of the energy received from the

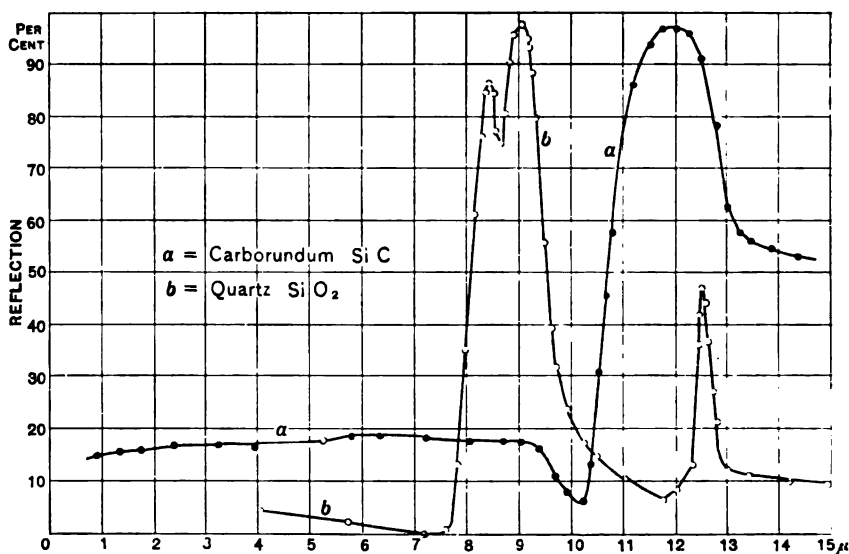


Fig. 17.

sun by the earth and by the moon, the part lying between the wave lengths 8 and 10μ will be reflected to a greater extent than for any other region of the spectrum. This means that in measuring the energy from the moon we must distinguish between selectively reflected energy from the sun and the direction radiation from the moon. The former has not heretofore been considered.

As is well known, absorption, reflection, and dispersion are intimately related. In the case of so-called "normal dispersion" the index of refraction decreases with increase in wave length. In the case of "anomalous dispersion," which occurs in the region of a

large absorption band, the index of refraction will be abnormally decreased on the side of the short wave lengths, while it will be abnormally increased on the side of the long wave lengths. Consequently, the reflection curve, which is a function of the refractive index, and the extinction coefficient will be abnormally increased on the side of the long wave lengths. This is well illustrated in the reflection curve of carborundum, SiC , Fig. 17, which is the most extraordinary substance yet examined. The reflecting power is usually high beyond 13μ , while in the region of 10μ it is suddenly decreased from a fairly constant value of 18 per cent to 6 per cent.

The reflecting power of the sulphides of zinc, iron, lead, and antimony are worthy of notice. They are known for their metallic lustre, especially stibnite, Sb_2S_3 . Their reflecting power in the

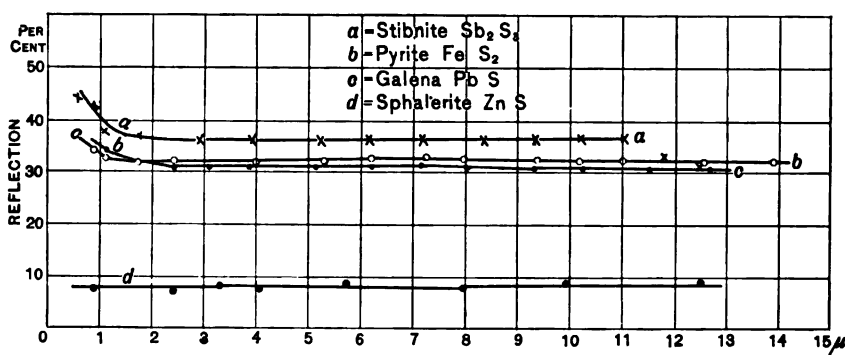


Fig. 18.

infra-red is equally conspicuous for its high value, Fig. 18, and among the most notable of these substances is stibnite.

The reflection curves of the sulphates of strontium, lead, barium, and calcium are shown along with the transmission curves of these substances, Figs. 13 and 14. Their study has demonstrated that the SO_4 ion has large absorption (reflection) bands at the harmonic intervals 4.55 and 9.1μ ; and the list of groups of atoms which have definite absorption bands in the infra-red may be extended so as to include the SO_4 group. The list consists of CH_2 , or CH_3 , NH_2 , C_6H_5 , NO , OH , NCS and SO_4 .

For the silicates the results as a whole show that there are no such definite bands, whether found by reflection or by absorption, as one might expect.

SUMMARY.

The transmission and reflection spectra of at least 120 compounds have been examined, many of them to $15\ \mu$, by means of a mirror spectrometer, a rock salt prism, and a Nichols radiometer. The aim of the investigation was the study of a series of minerals containing oxygen and hydrogen in the form of what is known as water of constitution and water of crystallization. The interpretation of the results is based upon the assumption that if the union of the oxygen and the hydrogen in the molecule is similar to that in water, then the absorption spectra of minerals containing these two elements thus united should show the absorption bands of water superposed upon the absorption spectrum of the other constituents. On the other hand, minerals containing oxygen and hydrogen as water of constitution should not show the water bands, except hydroxyl groups, which should show a band at $3\ \mu$.

The results show that of about 30 minerals containing "Water of crystallization" there are no important exceptions to the rule that they should show the bands of water. On the other hand the one important exception to the rule that minerals containing "water of constitution" should *not* show water bands is cane sugar. Minerals containing hydroxyl groups generally have a marked band at $3\ \mu$. Sulphates have a strong band at $4.55\ \mu$ and a less constantly recurring band at $9.1\ \mu$, due to the SO_4 ion. On the other hand silicates do not have such definite bands, which would seem to indicate that the structure of the silicate radical is different in each mineral containing that element.

The examination includes minerals of which the chemical constitution is in doubt; for example, talc and serpentine. The former is not supposed to contain hydroxyl groups while in the latter such groups are inferred. The present research supports these views in that the transmission curve of talc does not contain an absorption band at $3\ \mu$ while serpentine contains a large band at $3\ \mu$, which is in common with substances containing hydroxyl groups.

A VACUUM RADIOMICROMETER.

By W. W. Coblentz.

Radiometers having vanes weighing less than 10 mg are easily affected by earth tremors, so that for general work it is not advisable to reduce the weight much below this amount. Such a moving system is of course not very sensitive for a short period. With a fiber suspension of such a diameter that the maximum deflection is reached in 15 to 20 seconds, the sensitiveness is not much greater than 6 to 8 cm deflection per square millimeter surface exposed, the candle and scale each being at a distance of 1 meter. With this sensitiveness, however, the readings are always perfectly reliable, and such an instrument is to be preferred to a more sensitive one which gives longer deflections, but which, on account of lightness of the vane, is more affected by earth tremors and temperature changes.

On the other hand, in the radiomicrometer of Boys, the period is governed by the magnetic moment of the moving parts, rather than by the torsion of the fiber suspension, and in nearly all instruments thus far described it is much shorter than that of the type of radiometer just mentioned. Boys' radiomicrometer had a sensitiveness of only about 1 cm per sq. mm (candle and scale at 1 meter) for a suspension weighing 32 mg and a period of 10 seconds. He used a window to prevent air currents. This instrument has received but scant attention since it was first described by Boys, although Paschen¹ attempted to improve the instrument; but for the best junctions (out of perhaps 50) he could obtain only about 3 times the sensitiveness obtained by Boys, with a period of about 40 seconds. He then turned his attention to the bolometer with his well-known success.

¹ Paschen: *Ann. der Phys.* 48, p. 275; 1893.

The bolometer, however, with its delicate galvanometer requires an elaborate installation, and needs so much attention that the investigator's time is occupied principally with the care of the instrument, which should be a secondary matter in any work.

A great many radiometric investigations do not require the highest attainable sensitiveness and in such cases a radiometer or a radiomicrometer is a convenient laboratory accessory. The thermopile also has its place, but with this instrument heat conduction, local electromotive forces, etc., must be contended with.

The present study of the radiomicrometer was undertaken with the purpose of combining it with the radiometer, in order to develop more energy in the suspended system. The chief difficulty encountered in this combination is to obtain systems which are free from magnetic material. Diamagnetism also plays an important part in this form of thermojunction so that the radiometer effect is masked by the magnetic effect of the coil.

In the radiomicrometer proper the Bi-Sb junction hangs vertically at the axis of the system and therefore is not so seriously affected by its diamagnetism; hence, if the magnet is not too strong a fairly sensitive instrument is obtainable. Since a window improves its steadiness, the whole may as well be in a vacuum which eliminates heat conduction, as is well known for thermopiles. In this way the sensitiveness of a certain system, to be described presently, was increased by about 70 per cent. By using No. 40 copper wire and bits of Bi-Sb, soldered with Wood's alloy, it is possible to reduce the weight to 10 mg, which, as already mentioned, is a convenient weight. The period of such a system will be much less than that of the radiometer of similar sensitiveness.

Before describing the present instrument it will not be out of place to mention some facts concerning tests of sensitiveness of instruments. In previously described instruments no mention is made of the shielding of the thermojunctions from radiation other than that which falls directly upon the vane. The test of sensitiveness is hence not necessarily a fair one, since reflection from the walls has an enormous effect in increasing the sensitiveness. In fact, the measured sensitiveness may be twice the true value.

In the present instrument (Fig. 1), which is a design for a radiometer-radiomicrometer or a combination of the two, the receiv-

ing surface is directly behind the window, which is covered with an adjustable slit the length of the vane. In this manner no radiation reaches the walls of the instrument, which are black.

The bismuth for one part of the junction was obtained by melting

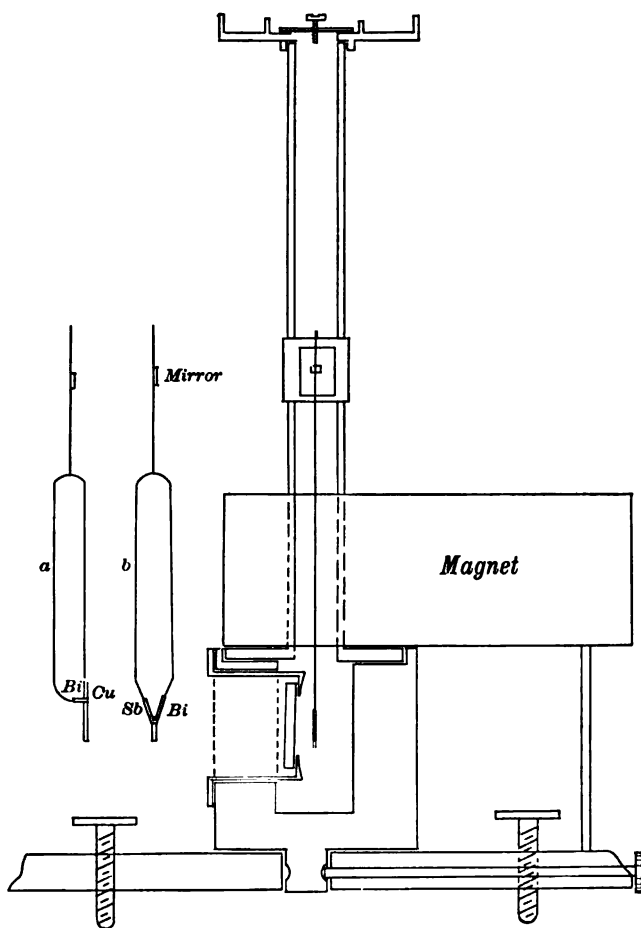


Fig. 1. *Vacuum Radiomicrometer*
Side View

the metal between glass or iron plates, which were then pressed together. These thin plates were then cut into strips from 0.15 to 0.2 mm wide. The antimony part of the junction was split from a large, well-crystallized piece of the metal. These pieces could not

be obtained quite so thin as the bismuth. The Bi and Sb pieces were then soldered, in the form of a Y, to a thin sheet of copper. This is not so good as to have the pieces more nearly parallel, since in the latter case the diamagnetic moment is reduced. The dimensions of the Bi and Sb pieces were about $3.5 \times 0.2 \times 0.1$ mm. The length of the copper loop of No. 40 wire was 4.5 cm. The area of the thin copper vane was 3 sq. mm. It was blackened by covering it with a little canada balsam, upon which was dusted fine copper oxide. The weight of the complete loop was less than 10 mg. The magnet was taken from a Weston ammeter. It was used without pole pieces, since the latter caused too much damping. The top was covered with the bulb of a thistle-tube, the mouth of which was ground flat. The heavy metal base of the radiomicrometer is of swedish iron. The narrow vertical tube is of brass. All the joints are made with Khotinsky cement, except the one at the top and the one containing the support of the rock salt window. The top seal is made with mercury. A torsion head is provided to bring the mirror to any desired position. The loop for the radiometer-radiomicrometer is shown in Fig. 1, *a*. The bismuth is about $2 \times 0.15 \times 0.1$ mm, soldered to a thin sheet of copper $8 \times 0.6 \times 0.1$ mm. In one case a sheet of mica of the same area was used and extended below the copper sheet which was then only 2 mm long. This loop also weighed less than 10 mg. As a radiometer the time required for a maximum deflection was 25 seconds, and its sensitiveness was about 3 to 4 cm per sq. mm area of vane, using a Geryk pump vacuum which was not high enough for maximum radiometer sensitiveness. This same loop used as a radiomicrometer had a sensitiveness of about 5 cm deflection per sq. mm area of vane, while its maximum deflection was reached in 8 seconds. The combination of these two was no better than the radiomicrometer, simply owing to the fact that the periods were different and the magnetic moment of the radiomicrometer masked the radiometer effect. Another loop twice as heavy had a half period (time of maximum steady deflection) of 5 seconds when used either as a radiometer or radiomicrometer. The actual sensitiveness was much lower than in the preceding loops. In this case the radiometer contributed only 15 per cent to the total deflection, while the sensitiveness of the

loop when used as a radiomicrometer was increased 50 per cent by placing it in a vacuum.²

In Fig. 1, *b*, is shown the loop for the radiomicrometer. The dimension of the Bi-Sb junction are indicated above. Its half period in air was 20 to 25 seconds and the deflections were 3.6 cm per sq. mm, for candle and scale at one meter. On exhausting the instrument with a Geryk pump the half period decreased to 12 or 14 seconds, while the deflection increased to 5.5 and 6 cm per sq. mm (70 per cent) of exposed area of vane; as the pumping progressed the sensitiveness increased. The result as a whole indicates that the weight of the moving coil can be reduced at least to one-third of those previously described and that its sensitiveness can be increased by at least 70 per cent by placing the loop in a vacuum. Since a window is used to increase the steadiness, the vacuum is no drawback, while, as already stated, it can be given a shorter period than a type of radiometer having a heavy vane.

The purpose in describing this form of instrument is not so much to show the sensitiveness of the present one as to indicate directions in which further improvements are possible. On account of the difficulties in preparing a nonmagnetic loop there is still much room for improvement in combining the radiometer and the radiomicrometer. For detecting electrical waves the system can be made still lighter and the heating arrangement can be brought close to the junction.

In conclusion, the fact may be emphasized that in the present test of sensitiveness only that radiation which passed through a slit and fell directly upon the vane was measured. If this precaution had not been taken the results would have been much higher and as in some other cases would have been misleading.

²The method of testing is as follows: The radiomicrometer effect is first found in air. It is then exhausted and the deflection again noticed. The magnet is then removed and the deflection obtained is due to the radiometer effect. The magnet must, of course, be placed so that the radiomicrometer deflection is in the same direction as that of the radiometer.

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